TRANSVERSE VIBRATIONS OF STUBBY CANTILEVER BEAMS

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INTRODUCTION

It has been customary, in the study of transverse vibrations of beams, to neglect the rotation which an element of the beam undergoes during its transverse displacement. Presumably this act is justified by the fact that the transverse dimension of the beam is small compared to its length.

The purpose of this paper is to present a study in which rotation is considered. The results indicate that the above approximation is valid, in general. In addition they show the first order correction which should be made in the event that the transverse dimension is not small compared to the length.

Results are given for the first normal mode of vibration only. This is done because the shortness of these beams makes the excitation of higher modes unlikely in practice.

DERIVATION OF THE EQUATIONS

Figure 1a illustrates a beam in transverse vibration. Its cross-section is shown in figure 1b.

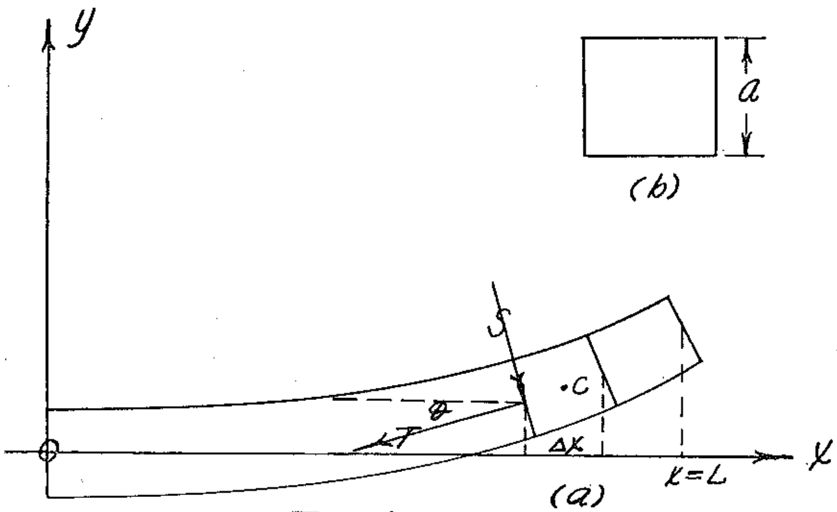

Fig. 1

FIGURE 1. TRANSVERSE VIBRATION IN BEAM

In figure 1a, T and S are the total tension and shear respectively on the left face of a beam element of length Δs. The vibrations to be considered are of such small amplitude that $\Delta s \cong \Delta x$, $\left(\text{i.e. } \left[\frac{\partial y}{\partial x}\right]^2 << 1\right)$. The point C, is the center-of-mass of the beam element.

Since there is assumed to be no motion in the x-direction

$$\frac{\partial}{\partial x}\{T\cos\theta - S\sin\theta\} = 0. \tag{1}$$

For motion in the y-direction one finds

$$\frac{\partial}{\partial x}\{T\sin\theta + S\cos\theta\} = \rho A\frac{\partial^2 y}{\partial t^2}. \tag{2}$$

If one considers the sum of the torques about an axis, through C, perpendicular to the paper he finds that

$$S + \int_A (z-\bar{z})\frac{\partial \tau}{\partial x}\,dA = \frac{\rho A a^2}{12}\frac{\partial^2\theta}{\partial t^2}, \tag{3}$$

in which τ is the tension stress over area dA on the left face of the element and A is the area of that face. From simple beam theory

$$\tau = E z\frac{\partial^2 y}{\partial x^2}, \tag{4}$$

in which E is Young's Modulus, and z is the distance of dA above the neutral plane. Equation 1 gives $T\cos\theta - S\sin\theta = 0$. (5)

Hence equation 2 becomes

$$\frac{\partial S}{\partial x} = \rho A\frac{\partial^2 y}{\partial t^2}. \tag{6}$$

This equation along with 3 and 4 gives

$$\rho A\frac{\partial^2 y}{\partial t^2} + \frac{\partial}{\partial x}\left\{EI\frac{\partial^3 y}{\partial x^3}\right\} = \frac{\partial}{\partial x}\left\{\frac{\rho A a^2}{12}\frac{\partial}{\partial x}\left(\frac{\partial^2 y}{\partial t^2}\right)\right\}, \tag{7}$$

since $\theta = \arctan\frac{\partial y}{\partial x} \cong \frac{\partial y}{\partial x}$. In equation 7, I is the second moment of the area A about a horizontal axis through C and in A.

In addition to equation 7, certain boundary conditions must be satisfied. They depend upon the facts that the beam is clamped at its left end and that at the right end is a free face. Thus at x = 0, both y and $\frac{\partial y}{\partial x} = 0$. At the free end both S and τ must vanish. Hence from equations 2, 4, and

$$EI\frac{\partial^3 y}{\partial x^3} = \frac{\rho A a^2}{12}\frac{\partial}{\partial x}\left(\frac{\partial^2 y}{\partial t^2}\right); \quad x = L, \tag{8}$$

and

$$\frac{\partial^2 y}{\partial x^2} = 0, \quad x = L. \tag{9}$$

III. SOLUTION OF THE EQUATIONS

Since we are interested only in normal modes it is convenient to set

$$y = u(x)e^{j\omega t}, \quad (j = \sqrt{-1}), \tag{10}$$

in which ω is the angular frequency of vibration, and u(x) is the time-independent amplitude of vibration. By the use of equation 10 the previously derived results can be expressed as follows

$$\left.\begin{aligned}
&\frac{d}{dx}\left\{EI\frac{d^3u}{dx^3}\right\} + \omega^2\frac{d}{dx}\left\{\frac{\rho A a^2}{12}\frac{du}{dx}\right\} - \rho A\omega^2 u = 0,\\
&\frac{d}{dx}\left\{EI\frac{d^2u}{dx^2}\right\} = -\frac{\rho A a^2}{12}\omega^2\frac{du}{dx}, \quad x = L,\\
&\frac{d^2u}{dx^2} = 0, \quad x = L.\\
&\frac{du}{dx} = 0, \quad u = 0, \quad x = 0.
\end{aligned}\right\} \tag{11}$$

Although equations 11 are valid for non-uniform beams the remainder of this paper will be concerned only with those which are uniform. As a consequence the differential equation could be handled by quadratures. A power series solution has been chosen, however, because of its utility for the visualization of effects of different orders.

For brevity let $\epsilon = \frac{\rho A\omega^2}{EI}$ and $\xi = \frac{a^2}{12}$. Then, the differential equation becomes

$$u^{[4]} + \xi\epsilon u^{[2]} - \epsilon u = 0. \tag{12}$$

Successive differentiations of equation 12 lead to

$$u^{[k]} + \xi\epsilon u^{[k-2]} - \epsilon u^{[k-4]} = 0. \tag{13}$$

Since we propose to express u as a Taylor's series in powers of (x - L) we shall need the derivations of u evaluated at x = L. We shall indicate this evaluation by the subscript, 1. Thus equation 13 becomes

$$u_1^{[k]} + \xi\epsilon u_1^{[k-2]} - \epsilon u_1^{[k-4]} = 0. \tag{14}$$

If one views equation 14 as a difference equation and makes use of the boundary conditions at x = L then he can show that

$$u_1^{[2s]} = \frac{\epsilon u_1}{\sqrt{\xi^2\epsilon^2+4\epsilon}}\left\{\alpha_1^{2(s-1)} - \alpha_2^{2(s-1)}\right\}, \tag{15}$$

and

$$u_1^{[2s+1]} = \frac{u_1^{[1]}}{\sqrt{\xi^2\epsilon^2+4\epsilon}}\left\{\alpha_1^{2(s+1)} - \alpha_2^{2(s+1)}\right\}, \tag{16}$$

in which $s = 0, 1, 2, 3, ----,$

$$\alpha_1 = \left\{\frac{-\xi\epsilon + \sqrt{\xi^2\epsilon^2+4\epsilon}}{2}\right\}^{\frac{1}{2}}, \tag{17}$$

and

$$\alpha_2 = \left\{\frac{-\xi\epsilon - \sqrt{\xi^2\epsilon^2+4\epsilon}}{2}\right\}^{\frac{1}{2}} \tag{18}$$

Thus by equations 15 and 16 it becomes possible to express u(x) as follows

$$u(x) = u_1 P_0 + u_1^{[1]} P_1, \tag{19}$$

in which P_0 and P_1 are infinite series in (x - L). In order to satisfy the boundary conditions at x = 0 we must set

$$u_1 P_0(0) + u_1^{[1]} P_1(0) = 0, \tag{20}$$

and

$$u_1 P_0^{[1]}(0) + u_1^{[1]} P_1^{[1]}(0) = 0. \tag{21}$$

The solution of 20 and 21 is trivial unless

$$\begin{vmatrix} P_0(0) & P_1(0) \\ P_0^{[1]}(0) & P_1^{[1]}(0) \end{vmatrix} = 0. \tag{22}$$

Since P_0 and P_1 are functions of ω, equation 22 provides the means for determining the normal mode angular frequencies.

The procedure employed was the solution of equation 22, by iteration for $a = 0$ followed by the assumption that for $a \neq 0$,

$$\omega^2 = \omega_0^2\left\{1 + \sigma_1 a^2 + \sigma_2 a^4 + \cdots\right\}, \tag{23}$$

for which the σ's were determined by a simple device.[1]

IV. RESULT AND CONCLUSIONS

The procedure outlined above was used to calculate ω^2 for the first normal mode. The result was

$$\omega^2 \cong 12.362 \left\{1 - \frac{0.449}{L^2} a^2\right\}. \qquad (24)$$

In order to determine the effect of rotation one can calculate the ratio a/L which gives rise to a percentage

$$p = 100 \frac{\omega_0^2 - \omega^2}{\omega_0^2} \qquad (25)$$

From equation 24 and 25

$$p = 44.9 \left(\frac{a}{L}\right)^2 \qquad (26)$$

Thus unless $a \geq L/3$ the error made by neglecting rotation will not be significant for most purposes.

BIBLIOGRAPHY

1. Cook, R. H., and Eatherton, L. J., Effect of Rotation' Upon the Normal Mode Frequencies of Transverse Vibration of a Cantilever Beam. Presented before Am. Math. Assoc., April (1954).