FACTORS AFFECTING THE FAILURE OF DUCTILE CYLINDRICAL SHELLS UNDER INTERNAL PRESSURE

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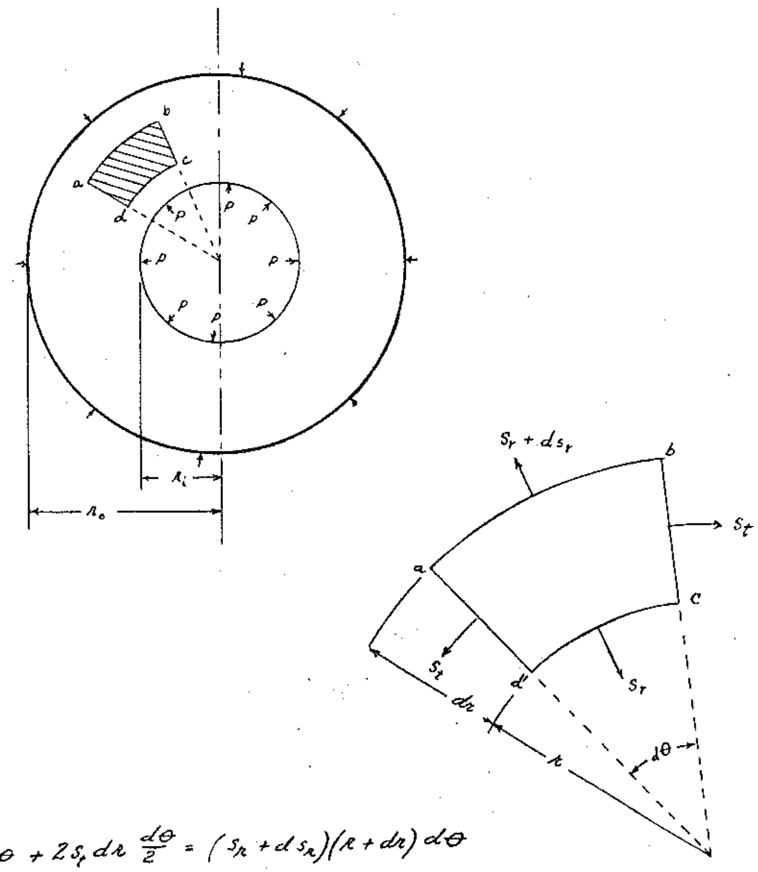
Customary design procedures postulate that for a given material a well defined stress quantity will result in failure. The magnitude of this stress is invariably determined by experimental means. Two such critical stress values are currently applied. The Elastic Limit or yield point stress is in vogue while the ultimate tensile stress as design criterion has found limited application. In both cases an empirically determined Factor of Safety relates the design stress to one of the above design criteria.

Either method is quite satisfactory for materials having an ultimate-to-yield stress ratio approaching unity. Also the method utilizing the yield point stress as design criterion is entirely reliable provided all strains remain elastic. Inasmuch as ultimate failure invariably is preceeded by plastic strains, it is obviously futile to attempt to apply this method in evaluating actual breaking loads. The ultimate stress design criterion is similarly lacking and particularly so for ductile materials, because it ignores the increase in stress resulting from the reduction of area prior to ultimate separation.

Both methods err on what is generally known as the "safe side"; hence the true Factor of Safety is larger than design computations would indicate. It has previously been shown by this author (1) that the design of cylindrical pressure vessels becomes critical as the wall ratio increases and that there is a limiting pressure which can be retained in a vessel of conventional design. It is evident that for economical design of pressure vessels it is necessary to establish a relationship between actual bursting pressure and the critical dimension of the vessels. Such is accomplished in the analysis which follows.

Also considered are the effects of bore strain and change of cross-sectional area prior to failure.



$$S_r\, r\, d\theta + 2S_t\, dr\, \frac{d\theta}{2} = (S_r + dS_r)(r+dr)\, d\theta$$

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$$S_t - S_r = r\,\frac{dS_r}{dr}$$

FIGURE 1. STRESSES ACTING ON CYLINDRICAL SHELL

Let area abcd represent an elemental area in the wall of a cylindrical shell, subjected to internal and/or external pressure. If s_t, s_r and s_a represent the tangential, radial, and axial stresses respectively at any point distant r from the axis, then the equilibrium equation for area abcd postulates as follows:

$$s_r\, r d\theta + 2s_t dr \frac{d\theta}{2} = (s_r + ds_r)(r+dr)\, d\theta$$

or, neglecting higher order differentials

$$s_t - s_r = r\,\frac{ds_r}{dr} \tag{1}$$

Also since s_t and s_r are principal stresses

$$s_t - s_r = 2s_s \tag{2}$$

Where s_s is the maximum shear stress.

It should be noted that the axial stress across the section has a constant magnitude and is in absolute value larger than s_r but smaller than s_t. Hence the conditions expressed by equations (1) and (2) are universally true so long as the shell remains cylindrical. They apply equally to a steel pipe or a bamboo rod, i.e., they are not confined to elastic conditions. It is particularly significant that they are applicable to ductile materials which have been strained beyond the elastic limit:

Solving equation (1) and (2) simultaneously, we have

$$\int ds_r = 2\int \frac{s_s}{r}\,dr \tag{3}$$

Integrating the LHS of equation (3) between the limits of the radial stress at the inner and outer boundaries of the shell we have

$$\int_{s_{ri}}^{s_{ro}} ds_r = s_{ro} - s_{ri} \tag{4}$$

For the case where internal pressure (P) only is acting we set $s_{ro} = 0$ and by inspection $s_{ri} = -P$

hence $\quad -s_{ri} = P \quad$ (5)

or the LHS of equation (3) is equivalent to the internal pressure.

The integration of the RHS of equation (3) is performed between the limits of the internal and external radii:

$$RHS = 2\int_{r_i}^{r_o} \frac{s_s}{r}\,dr \tag{6}$$

or

$$P = 2\int_{r_i'}^{r_o'} s_s\,d\,\log_e r \tag{7}$$

The limits of integration are primed to represent conditions after bore strain has taken place.

Equation (7) can be solved completely, provided the shear stress pattern across the section and the extended radii are known. However, the shear stress distribution in a plastic material does not yield readily to rigorous analysis. Also for purposes of this investigation, the evaluation of the bursting pressure (P_b) only is important. Ultimate failure is assumed to take place under the action of shear stress $s_{s\ ult}$ equivalent to ½ the ultimate tensile stress ($s_{s\ ult}$) of the material.

hence $$P_b = s_{ult} \log k' \tag{8}$$

where $$k' = \frac{r_o'}{r_i'}$$

The evaluation of the extended radii is accomplished on the assumption that the area of cross-section does not change, i.e., a constant density expansion is assumed.

Hence if e_i and e_o represent the radial strains of r_i and r_o respectively we can write

$$e_o^2 + 2r_oe_o = e_i^2 + 2\,r_i\,e_i \tag{9}$$

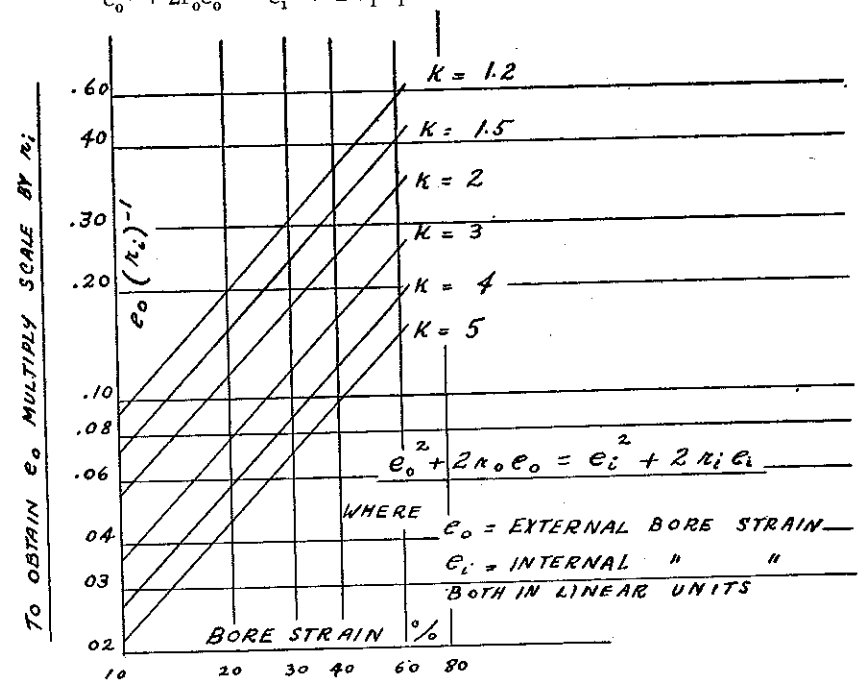

FIGURE 2. GRAPHICAL SOLUTION OF STRAIN EQUATION

The graph of Figure 2 represents the solution of this equation for materials suffering up to 70% bore strain at failure and for vessels having a wall ratio $\left(\frac{r_o}{r_i}\right)$ from 1.2 to 5. It should be noted that the ductility of the material will govern the amount of bore strain suffered by the vessel at failure. The higher the ductility of the material, the greater will be the strain and the corresponding reduction of area.

The radial strains (e_i and e_o) can be obtained from equation 9 or Figure 2 and strained diameter ratios (k′) are readily evaluated. The vessel geometry is such that $k' \angle k$.

Considering the effect of the reduction of diameter ratio on the ultimate stress it is evident that the actual ultimate stress is a function of $(k')^2$

hence

$$s_{actual} = \left(\frac{k}{K'}\right)^2 \quad s_{ult} \qquad (10)$$

and the Bursting pressure equation for cylindrical shells under internal pressure may be written thus:

$$P_b = \left(\frac{k}{K'}\right)^2 \; s_{ult} \quad \log_e K' \qquad (11)$$

It is suggested that the Bursting pressure as given above be used as a design criterion for pressure vessels and that Factors of Safety be applied against its value.

BIBLIOGRAPHY

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