# CRITICAL ROTATIONAL SPEEDS OF BULGED AND CONSTRICTED SHAFTS

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## INTRODUCTION

Figure 1 illustrates a shaft of circular cross-section which rotates with angular speed, $\omega$, about its line-of-center ,AA′.

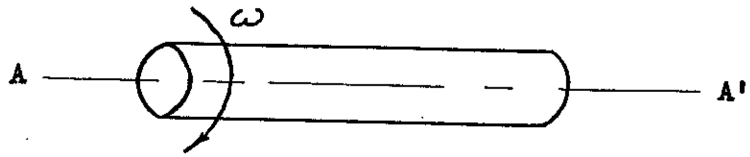
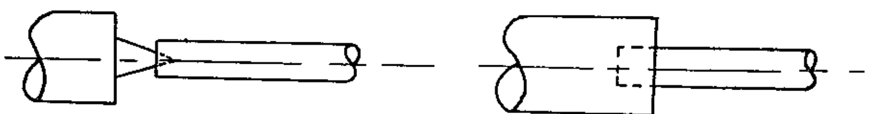



FIGURE 1. ROTATING SHAFT

In general, such shafts are supported in one of two ways, either by a "pinned" support or a "clamped" support. Figures 2a and 2b show these methods.



(a) Pinned (b) Clamped

FIGURE 2. TYPES OF SUPPORTS

The two ends may be supported in different ways.

If the shaft of Figure 1 is given a small displacement perpendicular to AA′ its subsequent motion depends upon $\omega$. If $\omega$ is less than a certain critical value, $\omega_c$, the shaft will oscillate about its line of centers. If $\omega$ is greater than $\omega_c$, the shaft will not oscillate but will undergo an increasing displacement and either be pulled from its supports or be destroyed.

The determination of $\omega_c$ is a characteristic-value problem for which the differential equation,

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - \rho\omega^2 Ay = 0 \qquad (1)$$

must hold. In equation 1, E is Young's modulus for the shaft material; I is the second moment of area of the cross-section about a diameter; $\rho$ is the density of the shaft material; A is the cross-sectional area; X is a space coordinate along the equilibrium line-of-centers of the shaft; and Y is the displacement of the actual line-of-centers from the equilibrium line-of-centers. If the shaft is of uniform cross-section I and A are constants,

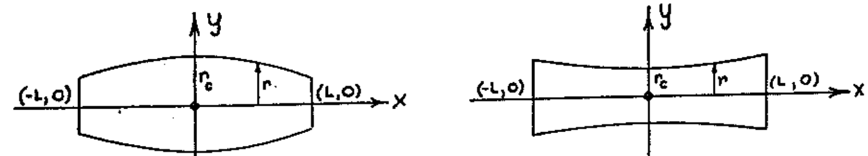


but if the shaft is non-uniform in cross-section then I and A and functions. of X.

The purpose of the present paper is to discuss a certain class of non-uniform shafts in order to determine the dependence of $\omega_c$ upon non-uniformity of cross-section.

## STATEMENT OF THE PROBLEM

Shafts of the types shown in figures 3a and 3b will be considered.




(a) Bulged Shaft ($\alpha$>0) (b) Constricted Shaft ($\alpha$<0)

FIGURE 3. TYPES OF SHAFTS

For the "bulged" shaft, as shown in Figure 3a, the radius, r, is a function, of x, given by $r = r_c\,(1 - \alpha x^2)$; $\alpha > 0$.
For the "constricted" shaft, as shown in Figure 3b, the radius, r, is a function, of x, given by $r = r_c\,(1 - \alpha x^2)$; $\alpha < 0$.
Such shafts will be discussed under two cases: Case I— Pinned Ends; Case II—Clamped Ends. The mathematical description of each of these is given below.

### Case I—Pinned Ends

$$\frac{d^2}{dx^2}\left\{(1-\alpha x^2)^4\frac{d^2y}{dx^2}\right\} - \lambda^2(1-\alpha x^2)^2 y = 0, \qquad (2)$$

$$y(x) \text{ symmetric to } y\text{-axis}, \qquad (3)$$

$$\frac{d^2y}{dx^2} = 0 \quad \text{at} \quad x = -L, \qquad (4)$$

$$y = 0 \quad \text{at} \quad x = -L. \qquad (5)$$

**Case II—Clamped Ends**

$$\frac{d^2}{dx^2}\left\{(1-\alpha x^2)^4\frac{d^2y}{dx^2}\right\} - \lambda^2(1-\alpha x^2)^2 y = 0 , \qquad (6)$$

$$y(x) \text{ symmetric to } y\text{-axis}, \qquad (7)$$

$$\frac{dy}{dx} = 0 \quad \text{at} \quad x = -L , \qquad (8)$$

$$y = 0 \quad \text{at} \quad x = -L , \qquad (9)$$

$$\lambda^2 = \frac{2\rho\omega^2}{E r_c^2} . \qquad (10)$$

Values of $\omega_c$ are to be calculated for a small range of values in the neighborhood of $\alpha = 0$.

## METHOD

The method used was suggested by Cook (1, 2), and later discussed (3) by him. It consists, essentially, of the solution of equations (2) and (6) as a Maclaurin series by successive differentiation. The boundary conditions are then imposed and lead to a determinant equation from which $\omega_c$ can be found. We shall present the solution of Case I in detail, and give the calculated results for both cases.

Let

$$g = (1-\alpha x^2)^4 \quad \text{and} \quad h = (1-\alpha x^2)^2 \qquad (11)$$

Then

$$\left.\begin{array}{l} g(0) = 1 ;\quad g^{[1]}_{(0)} = 0 ;\quad g^{[2]}_{(0)} = -8\alpha ;\quad g^{[3]}_{(0)} = 0 ; \\ g^{[4]}_{(0)} = 6\cdot 4!\alpha^2 ;\quad g^{[5]}_{(0)} = 0 ;\quad g^{[6]}_{(0)} = -4\cdot 6!\alpha^3 ;\quad g^{[7]}_{(0)} = 0 ; \\ g^{[8]}_{(0)} = 8!\alpha^4 . \end{array}\right\} \qquad (12)$$



and

$$\left.\begin{array}{l} h(0) = 1 ;\quad h^{[1]}_{(0)} = 0 ;\quad h^{[2]}_{(0)} = -4\alpha \\ h^{[3]}_{(0)} = 0 ;\quad h^{[4]}_{(0)} = 4!\alpha^2 . \end{array}\right\} \qquad (13)$$

If we put $g^{[K]}_{(0)} = g_{K0}$ and $h^{[K]}_{(0)} = h_{K0}$ we can express any even derivative of the left member of (2) in simple form for $x = 0$. All odd derivaties of $y_{(x)}$ are zero at $x = 0$, because of condition (3). Thus

$$g_{0,0}\, y_0^{[2n+4]} + \binom{2n+2}{2} g_{2,0}\, y_0^{[2n+2]} + \sum_{S=1}^{n}\left\{\binom{2n+2}{2S-2} g_{2n+4-2S,0} - \lambda^2\binom{2n}{2S} h_{2n-2S,0}\right\} y^{[2S]} - \lambda^2 h_{2n,0}\, y_0 = 0 \quad (n = 1, 2, 3, \cdots) . \qquad (14)$$

In equation (14) the symbol $\binom{r}{K}$ is a binomial coefficient given by

$$\binom{r}{K} = \frac{r!}{K!(r-K)!} . \qquad (15)$$

Equation (2) becomes

$$g_{0,0}\, y_0^{[4]} + g_{2,0}\, y_0^{[2]} - \lambda^2 h_{0,0}\, y_0 = 0 . \qquad (16)$$

Equations (12), (13), (14) and (16) can be combined to give

$$\left.\begin{array}{l} y_0^{[4]} = \lambda^2 y_0 + 8\alpha y_0^{[2]} , \\ y_0^{[6]} = 44\alpha\lambda^2 y_0 + (240\alpha^2 + \lambda^2)\, y_0^{[2]} , \\ y_0^{[8]} = (3144\alpha^2\lambda^2 + \lambda^4)\, y_0 + (14400\alpha^3 + 104\alpha\lambda^2)\, y_0^{[2]} , \end{array}\right\} \qquad (17)$$

etc.

If equations (17) are substituted into the Maclaurin expansion the result is

$$y = y_0 \left\{ 1 + \lambda^2 \left( \frac{x^4}{4!} + 44\alpha \frac{x^6}{6!} + 3144\alpha^2 \frac{x^8}{8!} + \cdots \right) + \lambda^4 \left( \frac{x^8}{8!} + \cdots \right) + \cdots \right\} +$$

$$y_0^{[2]} \left\{ \left( \frac{x^2}{2!} + 8\alpha \frac{x^4}{4!} + 240\alpha^2 \frac{x^6}{6!} + 14400\alpha^3 \frac{x^8}{8!} + \cdots \right) + \right.$$

$$\left. \lambda^2 \left( \frac{x^6}{6!} + 104\alpha \frac{x^8}{8!} + \cdots \right) + \cdots \right\} . \qquad (18)$$

If conditions (4) and (5) are imposed upon (18) the result is

$$\begin{vmatrix} F_{11}(\lambda^2 L^4, \alpha) & F_{12}(\lambda^2 L^4, \alpha) \\ F_{21}(\lambda^2 L^4, \alpha) & F_{22}(\lambda^2 L^4, \alpha) \end{vmatrix} = 0 . \qquad (19)$$

Equation (19) can be solved for $\lambda^2 L^4$ in terms of $\alpha$ and thus $\omega_c$ can be evaluated.

## RESULTS

The case which was considered in the previous section (Pinned Ends) leads to the results shown in Table I.

**TABLE I. RESULTS FOR PINNED ENDS ($r_e$ = radius of an end)**

| $\alpha$ | $\omega_c$ |
|---|---|
| $\frac{.1}{L^2}$ | $\frac{2.69}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $\frac{.01}{L^2}$ | $\frac{2.47}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $0$ | $\frac{2.45}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $-\frac{.01}{L^2}$ | $\frac{2.42}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $-\frac{.1}{L^2}$ | $\frac{2.18}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |



The same procedure can be used for Case II, Clamped Ends, except that condition 8 replaces condition 4. The results for this Case are shown in Table II.

**TABLE II.**

**Results for Clamped Ends ($r_e$ = radius of an end)**

| $\alpha$ | $\omega_c$ |
|---|---|
| $\frac{.1}{L^2}$ | $\frac{6.55}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $\frac{.01}{L^2}$ | $\frac{5.49}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $0$ | $\frac{5.48}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $-\frac{.01}{L^2}$ | $\frac{5.46}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |
| $-\frac{.1}{L^2}$ | $\frac{5.35}{L^2}\left[\frac{E r_e^2}{\rho}\right]^{\frac{1}{2}}$ |

We note that for Case I a ten percent change in taper (i.e. a change of ten percent in $\alpha$ ) leads to a change of roughly ten percent in $\omega_c$. For Case II, however, a ten percent change in taper leads to not more than a 2.5% change in $\omega_c$.

## BIBLIOGRAPHY

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3. Cook, R. H., The Method of Frobenius. Unpublished Paper (1953).