EXPANSION OF TYPE FORMS INTO FOURIER SERIES

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The theory of Fourier series is one of the most important developments of analysis, and it serves as an indispensable instrument in the treatment of nearly every physical problem. Solution of such important problems as sound vibration, propagation of electrical currents and radio waves, heat conduction, and mechanical vibrations give but a mere indication of its value.

It was in 1822, that Joseph Fourier presented his papers before the French Academy on the theory and flow of heat. He presented the absolutely revolutionary doctrine that an arbitrarily given curve or function, irrespective of its nature, could be represented in any interval by a trigonometric series. Fourier sought no strict proof for his assertion, but the examples given by him upheld his assertions.

In 1829, Dirichlet brought forth his famous set of sufficient conditions for the expansion of a function into a Fourier series. These are now known as the **Dirichlet conditions.** Since then vast amounts of literature have grown around the subject of Fourier series.

A function satisfying the Dirichlet conditions in the interval $(-\pi, \pi)$, may be expanded into the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the coefficients a_0, b_n, and a_n are given by the formulas

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Trigonometric series are capable of representing extremely general types of functions and so constitute a far more powerful instrument of analysis than power series. Some of these general type expansions were obtained and a few of the unique results when specific values were substituted for the arbitrary constants, **a**, **b**, and **c**.

The following expansions were developed from the general quadratic, $f(x) = ax^2 + bx + c$:

$$1. \quad f(x) = \frac{2a}{\pi}\left[\left(\frac{\pi^2}{(2n-1)} - \frac{4}{(2n-1)^3}\right)\sum_{n=1}^{\infty} \sin(2n-1)x - \frac{\pi^2}{2n}\sum_{n=1}^{\infty} \sin 2nx\right] + 2b\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n} + \frac{4c}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$$

with the interval of $0 < x < \pi$.

$$2. \quad f(x) = \frac{a\pi^2}{3} + \frac{b\pi}{2} + c - 4a\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2} - \frac{4b}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{2n-1}$$

with the interval of $0 \leq x \leq \pi$.

The following expansions were developed from the two trigonometric forms, a sin bx and a cos bx, where **b** is not an integer:

$$3. \quad a \sin bx = \frac{2a \sin b\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n \sin nx}{n^2 - b^2}$$

in the interval $0 < x < \pi$

$$4. \quad a \cos bx = \frac{2ab \sin b\pi}{\pi}\left[\frac{1}{2b^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2 - b^2}\right]$$

in the interval $0 \leq x \leq \pi$.

The following expansions were developed as special cases of the logarithm of the trigonometric forms, log a sin bx and log a cos bx:

5. $$\log 2 \sin \frac{x}{2} = -\sum_{n=1}^{\infty} \frac{\cos nx}{n}$$

in the interval of $0 < x \leq \pi$.

6. $$\log 2 \cos \frac{x}{2} = -\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n}$$

in the interval of $0 \leq x < \pi$.

The following expansion was developed from the general exponential term e^z, where $z = x + iy$, the complex number:

7. $$e^z = \frac{4}{n} \sum_{n=2}^{\infty} \frac{n \sin nx \sin ny}{n^4 - 1^4} \left[e^{\pi + i\pi} + e^{\pi}(-1)^{n+1} + e^{i\pi}(-1)^{n+1} + 1 \right]$$

in the interval of $0 < x < \pi$

As each general expansion was obtained, it was necessary to prove that the Fourier series was convergent and equal to the function as n became infinite. This was accomplished with the aid of the **Dirichlet integral** and the **Dirichlet conditions.**

It was noticed that when certain values were assigned to the arbitrary constants and the value of x was fixed, the Fourier series turned into a power series. Further investigation of these power series showed that here was a unique method for determining several identities.

When the general quadratic is treated in such a manner, then values for $\frac{\pi}{4}$, $\frac{\pi^2}{8}$, and $\frac{\pi^2}{24}$ may be found as follows:

(a) Let $a = 0$, $b = 0$, $c = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$, then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots;$$

(b) Let $a = 0$, $b = -1$, $c = \frac{\pi}{4}$, and $x = 0$, then

$$\frac{\pi^2}{8} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots;$$

(c) Let $a = -\frac{\pi}{8}$, $b = \frac{\pi}{8}$, $c = 0$, and $x = 0$, then

$$\frac{\pi^2}{24} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots .$$

In the expansion of $a \cos bx$, let $x = \pi$, $b = x$, and integrate, then

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right),$$

the infinite product for $\sin \pi x$ may be found. In a similar manner when $x = 0$ and $b = x$,

$$\cos \pi x = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2}\right)$$

the infinite product for $\cos \pi x$ may be found.

By letting $a = 2$, $b = 1$, and $x = \pi$ in the expansion of $\log a \sin bx$, the value for $\log 2$ may be found to be

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

When the real part of the exponential expansion is allowed to go to zero, then Euler's important formulas

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \qquad \cos y = \frac{e^{iy} + e^{-iy}}{2}$$

may be verified from the resulting expansions and those obtained from the trigonometric forms.

The developments given here are but a few of the more interesting expansions which may be obtained merely by substituting certain values for the arbitrary constants and the variable x. It is believed that further investigations of these general expansions will yield new ways of determining many of the general identities.