WHEN RELATIVELY ALL RELATIVE PRIMES ARE PRIME

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An integer greater than 1 is called *prime* if it has no divisors in the positive integers besides itself and 1. Two positive integers are called *relatively prime* if they share no divisors besides 1. Some positive integers n have the property that all the integers between 1 and n which are relatively prime to n are prime. For brevity, we will say that such n have *Property P*. Among the first twenty positive integers, 1 and 2 have Property P by default, and 3, 4, 6, 12, and 18 have Property P for more interesting reasons, while the rest fail Property P.

Schatunovsky's Problem. Is there a largest integer with Property P, and if so, what is it?

Solution. Let p_k denote the kth prime. ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.)

(1) If $(p_k)^2 \le n < (p_{k+1})^2$ and $p_1 p_2 \cdots p_k > (p_{k+1})^2$, then n does not have Property P.

Proof. $p_1 p_2 \cdots p_k > (p_{k+1})^2 > n$. Therefore $p_1 p_2 \cdots p_k$ is not a divisor of n. Hence one of $p_1, p_2, \ldots, p_k$, say p_j, is not a divisor of n. Now $(p_j)^2$ is smaller than n, and relatively prime to n, but not prime.

(2) If $p_1 p_2 \cdots p_k > (p_{k+1})^2$, then $p_1 p_2 \cdots p_m > (p_{m+1})^2$ for all $m \ge k$.

Proof. Use induction on m. If $p_1 p_2 \cdots p_m > (p_{m+1})^2$, then $p_1 p_2 \cdots p_{m+1} > 4\,(p_{m+1})^2$ because $p_{m+1} > 4$, and $p_1 p_2 \cdots p_{m+1} > (p_{m+2})^2$ because $p_{m+2} < 2\,p_{m+1}$.

Conclusion. From (1) and the fact that $2 \cdot 3 \cdot 5 \cdot 7 > 11^2$, we know that 49 does not have Property P, nor does anything between 49 and 121. From (2) it follows that all integers greater than or equal to 49 fail Property P. Thus there is a largest integer which has Property P, and it is less than 49. Checking backward from 49, one finds the number is 30.

A Variation. What is the largest odd integer n such that all of the odd integers between 1 and n which are relatively prime to n are prime?

This problem was posed in a recent issue of the American Mathematical Monthly. The answer, obtained by the same methods as used above, is 105.

A Generalization. A positive integer n has *Property P(m)* if n is not divisible by m and all the integers between 1 and n which are relatively prime to n and not divisible by m are prime. What is the largest integer that has Property P(m) ?

Partial Solution. Suppose m is prime, say $m = p_j$. Let k be the smallest integer such that $p_1 p_2 \cdots p_{j-1} p_{j+1} \cdots p_k > (p_{k+1})^2$. Then the largest integer having Property P(m) is the largest multiple of $p_1 p_2 \cdots p_{j-1} p_{j+1} \cdots p_{k-1}$ which is smaller than $(p_k)^2$. For $m = 2$ this yields 105, as stated above. For $m = 3$ the result is 70, and for $m = 5$ it is 84. For all primes $m \ge 7$, the answer is 30, the number which turned up for the original problem.

The Remainder. What is the largest integer having property P(m) when m is not prime?

References

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