RECURRENCE RELATIONS AND THE POWERS OF DIAGONIZABLE MATRICES

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A recurrence relation for a sequence of numbers is a formula that describes each member of the sequence in terms of previous members. Recurrence relations are very important in mathematics and in its applications to such diverse areas as physics, computer science, statistics, botany, economics, psychology, and sociology.

Suppose that a sequence of numbers $U_0, U_1, U_2, U_3, \ldots$ is given by the recurrence relation

$$U_k = U_{k-1} + U_{k-2} \quad \text{for all } k \geq 2 \text{ and } U_0 = U_1 = 1 .$$

Using this formula to compute the first twelve terms yields $1,1,2,3,5,8,13,21,34,55,89,144$.

The sequence $1,1,2,3,5,8,13,\ldots$ is called the *Fibonacci sequence*, and the numbers in the sequence are called *Fibonacci numbers*. These numbers occur in the most remarkable places in nature. For example, on some plants, thorns and leaves grow in a spiral pattern, and the number of growths per revolution about the stalk is a ratio of two Fibonacci numbers. The apple tree and the oak tree have five growths for every two turns around the stalk, the pear tree has eight growths for every three turns, and the willow tree has thirteen growths for every five turns.

Computing the first dozen terms from the recursion formula was easy. However, to compute the 900th term of the sequence it is necessary to know the 898th and 899th terms. The computation of these terms using the recurrence relation is a very long and laborious task. The computation of the kth term of the sequence can be simplified greatly by using powers of the diagonizable matrix which corresponds to the recurrence relation.

The first step in finding an explicit formula for the kth Fibonacci number is to write the recurrence relation $U_k = U_{k-1} + U_{k-2}$ in matrix form. This can be accomplished simply by writing

$$M_k = A\, M_{k-1}$$

where $M_k = \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

This formula is correct because

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_k \\ U_{k-1} \end{pmatrix} = \begin{pmatrix} U_k + U_{k-1} \\ U_k \end{pmatrix},$$

so that $M_k = A\, M_{k-1}$ says $U_{k+1} = U_k + U_{k-1}$, which is the same as $U_k = U_{k-1} + U_{k-2}$, and $U_k = U_k$, which of course is true in any case.

An immediate consequence is the general formula

$$M_k = A^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{that is,} \qquad \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

by which U_{k+1} can be calculated for any $k \geq 0$. Furthermore, this is not a recursion formula, as there are no U_j on the right side of the equation.

Nevertheless, calculating

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k$$

is still a laborious task. In fact, it is really a recursive job, because for matrices A in general, we calculate A^k by repeated multiplication. For *diagonal* matrices, that is, matrices with all zero entries off the principal diagonal, raising to the kth power is easy: simply raise each entry to the kth power. Thus we can alleviate the difficulty of calculating A^k by diagonalizing A.

Our technique of diagonalizing A involves *eigenvalues* and *eigenvectors*. For an $n \times n$ matrix A, the number λ is an eigenvalue if and only if there is a non-zero n-dimensional vector $\mathbf{v}$ for which

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Any such vector $\mathbf{v}$ is called an eigenvector of A associated with the eigenvalue λ.

The equation $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to

$$(A - \lambda I)\, \mathbf{v} = \mathbf{0}$$

where I is the $n \times n$ identity matrix and $\mathbf{0}$ is the n-dimensional zero vector. Now $(A - \lambda I)\, \mathbf{v} = \mathbf{0}$ is nothing more than a homogeneous system of n linear equations in n variables. Thus the eigenvalues of the matrix A are those numbers λ for which the homogenous system has a non-zero solution, and the eigenvectors of A associated with λ are the non-zero vectors such that $(A - \lambda I)\, \mathbf{v} = \mathbf{0}$. The system $(A - \lambda I)\, \mathbf{v} = \mathbf{0}$ has a non-zero solution if and only if its coefficient matrix $A - \lambda I$ is non-invertible, and the matrix $A - \lambda I$ is non-invertible if and only if its determinant is equal to zero.

Now we determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\det\,(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1$$

Therefore the eigenvalues of A are the solutions of $\lambda^2 - \lambda - 1 = 0$, namely $\lambda_1 = \dfrac{1 + \sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1 - \sqrt{5}}{2}$.

$$\text{Now } (A - \lambda_1 I)\, \mathbf{v} = \mathbf{0} \iff \begin{pmatrix} \frac{1 - \sqrt{5}}{2} & 1 \\ 1 & \frac{-1 - \sqrt{5}}{2} \end{pmatrix} \mathbf{v} = \mathbf{0}$$

$$\iff \begin{pmatrix} 1 & \frac{-1 - \sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0} \iff \mathbf{v} = c \begin{pmatrix} \frac{1 + \sqrt{5}}{2} \\ 1 \end{pmatrix} \text{ for some } c.$$

Therefore

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

is an eigenvector for A corresponding to the eigenvalue λ_1.

Similarly $(A - \lambda_2 I)\,\mathbf{v} = 0 \iff \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix} \mathbf{v} = 0$

$$\iff \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix} \mathbf{v} = 0 \iff \mathbf{v} = c \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \text{ for some } c.$$

Therefore

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

is an eigenvector for A corresponding to the eigenvalue λ_2.

Now we can diagonalize A using

$$P = \left(\mathbf{v}_1 \mid \mathbf{v}_2 \right) \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

This works because $P\,D = \left(\lambda_1 \mathbf{v}_1 \mid \lambda_2 \mathbf{v}_2 \right) = \left(A\mathbf{v}_1 \mid A\mathbf{v}_2 \right) = A\,P$, yielding

$$A = P\,D\,P^{-1}.$$

The application we wish to make is

$$A^k = (P\,D\,P^{-1})^k = (P\,D\,P^{-1})\,(P\,D\,P^{-1})\,(P\,D\,P^{-1}) \cdots (P\,D\,P^{-1}) = P\,D^k\,P^{-1}.$$

Therefore to compute the kth power of A, all we need do is compute

$$D^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$

Calculating A^k explicitly,

$$A^k = P\,D^k\,P^{-1}$$

$$= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^k \end{bmatrix} \left(\frac{1}{\sqrt{5}}\right) \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}.$$

Substituting this into $M_k = A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, it follows that

$$\begin{bmatrix} U_{k+1} \\ U_k \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \end{bmatrix}.$$

By equating the corresponding entries of the above matrices we derive a formula for the kth Fibonacci number:

$$U_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \right] \quad \text{for all } k \geq 0.$$

It is interesting that although the kth Fibonacci number U_k is an integer, the formula for U_k contains some complicated irrational numbers involving the square root of 5.

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