THE RICHARDSON ARMS RACE MODEL IN RECURSION EQUATIONS

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The system of two first-order linear differential equations for the two-nation arms race model extends for three nations to the following form.

$$\frac{dx}{dt} = (ay + bz) - mx + g$$

$$\frac{dy}{dt} = (cx + dz) - ny + h$$

$$\frac{dz}{dt} = (ex + fy) - pz + i$$

The table below gives the meanings of the variables and parameters.

	Nation X	Nation Y	Nation Z
War Potential Time Dependent Variables	$x(t)$	$y(t)$	$z(t)$
Rates of Change (Deviations)	$\frac{dx}{dt}$	$\frac{dy}{dt}$	$\frac{dz}{dt}$
Dependency on Other Nations	$ay + bz$	$cx + dz$	$ex + fy$
Restraining Influences	$-mx$	$-ny$	$-pz$
Underlying Grievances	g	h	i

Following Zinnes, Gillespie, and Tahim for the two-nation model, a discrete interpretation of the model for three nations takes the form of coupled first-order difference or recursion equations.

$$x(k) = -m\,x(k-1) + a\,y(k-1) + b\,z(k-1) + g$$
$$y(k) = c\,x(k-1) - n\,y(k-1) + d\,z(k-1) + h$$
$$z(k) = e\,x(k-1) + f\,y(k-1) - p\,z(k-1) + i$$

These three recursion equations are repeated by relating war potential variables for the discrete time k-1 to time k-2, and discrete time k-2 to time k-3. The nine recursion relations provide the means to obtain a third-order difference equation for x.

$$\begin{aligned} x(k) = & - (m + n + p)\, x(k-1) \\ & + (ac - mp + be + df - np - mn)\, x(k-2) \\ & + (ade + acp - mnp + mdf)\, x(k-3) \\ & + (\text{a constant involving } g,\ h,\ i, \text{ and the nine other parameters}) \end{aligned}$$

For stability considerations, write the above third-order difference equation as

$$x(k) = Q\, x(k-1) + R\, x(k-2) + S\, x(k-3) + C\ ,$$

introduce a stability parameter ρ with the simple first-order difference relation

$$x(k) = \rho\, x(k-1)\ ,$$

and assume an initial condition

$$x(0) = x_o\ .$$

From these relations there follows the cubic equation in ρ

$$\rho^3 - Q\rho^2 - R\rho - S = 0\ .$$

The roots of this equation signify the stability of the arms race. If all roots have absolute values less than one a stability exists, but just one root with absolute value greater than one indicates instability.

An equilibrium point involving all three nations means that

$$x(k) - x(k-1) = 0\ ,$$

$$y(k) - y(k-1) = 0\ ,$$

and

$$z(k) - z(k-1) = 0.$$

In matrix form such an equilibrium point is given by the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = - \begin{pmatrix} -m-1 & a & b \\ c & -n-1 & d \\ e & f & -p-1 \end{pmatrix}^{-1} \begin{pmatrix} g \\ h \\ i \end{pmatrix}.$$

REFERENCES

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