TRIANGULAR NUMBERS WITH REPEATED DIGITS

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A triangular number is a positive integer of the form $T_n = n(n+1)/2$ where n is a positive integer.

E. B. Escott (1905) proved that 1, 3, 6, 55, 66 and 666 are the only triangular numbers of less than 30 digits that consist of a single repeated digit. This paper will complete the proof of this theorem and show that there are no other triangular numbers of any digit length which consist only of a single repeated digit. Formally, we have the following theorem.

Theorem: The triangular numbers 1, 3, 6, 55, 66 and 666 are the only triangular numbers consisting of a single repeated digit.

Proof: If a triangular number, T_k, is to consist of a digit d repeated j-1 times, we must have

(1) $$\frac{k(k+1)}{2} = d(10^j - 1)/9.$$

Solving for k, we have

(2) $$k = (-9 \pm (81 + 8d(10^j - 1))^{1/2})/18.$$

For such a k to exist, it is necessary that

(3) $$n = 1 + 8d(10^j - 1)/9$$

be a perfect square.

Since d is the digit which will be repeated, it must be an integer from 1 to 9. We first note that the digits 2, 4, 7, 9 can be eliminated as possibilities. This is seen from the following table:

n(mod 10)	n^2(mod 10)	$\frac{n^2+n}{2}$(mod 10)
0	0	0
1	1	1
2	4	3
3	9	6
4	6	5
5	5	5
6	6	6
7	9	8
8	4	6
9	1	5

It is therefore clear that 2, 4, 7, 9 cannot occur as a last digit of a triangular number and so cannot be repeated. Now we consider in turn d = 1, 3, 5, 6, 8.

If d = 1 is repeated, then for various values of j, equation 3) is calculated to be a number of the form 88 ··· 889 (the dots mean that the digit 8 is repeated) or 9. That is, if n is a perfect square, we have a triangular T_k which has repeated digits and can be calculated from equations 1) and 2). From n = 9 we get $T_1 = 1$.

If d = 3, the same type of analysis shows that n is of the forms 25 or 266 ··· 665.

If d = 5, n is of the form 44 ··· 441.

If d = 6, n is of the forms 49, 529 or 533 ··· 329.

If d = 8, n is of the forms 705 or 711 ··· 105.

We see that

n = 9 gives $T_1 = 1$
n = 25 gives $T_2 = 3$
n = 49 gives $T_3 = 6$
n = 441 gives $T_{10} = 55$
n = 529 gives $T_{11} = 66$
n = 5329 gives $T_{36} = 666$.

We will now show that no other possibility for n gives a perfect square. Our method is as follows.

Assume 88 ··· 89 is to be a square, say z^2. Then in decimal notation,

$$z = d_n d_{n-1} \cdots d_2 d_1 d_0.$$

Since 88 ··· 89 end in 9, d_0 must be 3 or 7. If d_0 is 3, then to get the last two digits of 88 ··· 89 to be 89, d_1 must be 3 or 8. If d_1 is to be 8, then from the form of 88 ··· 89, d_2 must be 5. Now there is no choice of d_3 such that $(d_3 583)^2$ has its last four digits of form 8889. Thus this chain stops. Now d_1 was 3 or 8 and the choice of 8 leads to a contradiction, so we keep $d_0 = 3$ and choose $d_1 = 3$. Then d_2 must be 8 which again leads to a contradiction. We have thus eliminated all possibilities where d_0 was 3. But there were two choices for d_0 so we choose $d_0 = 7$ and begin the same analysis.

We represent the procedure just outlined with the following notation

3 - 8 - 5 - *.

The 3 is the initial choice for d_0; the digit 8 is one of the possibilities for d_1; 5 is a possibility for d_2; * means that there is no acceptable digit for d_3 so as to make the last four digits of $(d_3 583)^2$ of the correct form.

Thus to obtain a number of the form $88 \cdots 89 = z^2$, the possibilities for 2 are:

3 - 3 - 8 - *
3 - 8 - 5 - *
7 - 1 - 9 - 7 - *
7 - 1 - 9 - 2 - 2 - *
7 - 1 - 9 - 7 - 5 - *
7 - 1 - 4 - *
7 - 6 - 1 - *
7 - 6 - 6 - 1 - *
7 - 6 - 6 - 6 - 1 - *
7 - 6 - 6 - ··· - 1 - *.

Thus all possibilities are eliminated and no number of the form $88 \cdots 889$ can be a perfect square, and $T_1 = 1$ is the only triangular with all digits being 1.

If $266 \cdots 665$ is to be a perfect square, then it must be 25. For if a 6 or series of 6's occurs between the 2 and 5, there is no possibility for the d_1 position of z. Hence the only triangular having all digits 3 is $T_2 = 3$.

If $44 \cdots 441$ is to be a square, it must be 441. The possibilities for more than two 4's are listed:

1 - 2 - 5 - *
1 - 7 - 2 - *
1 - 7 - 7 - 5 - 2 - *
1 - 7 - 7 - 5 - 7 - *
9 - 2 - 2 - 4 - 2 - 3 - 5 - *
9 - 2 - 2 - 4 - 2 - 8 - *
9 - 2 - 2 - 4 - 7 - *
9 - 2 - 2 - 9 - *
9 - 2 - 7 - *
9 - 7 - 4 - *
9 - 7 - 9 - 2 - *
9 - 7 - 9 - 7 - 6 - *
9 - 7 - 9 - 7 - 1 - 4 - 1 - 6 - *
9 - 7 - 9 - 7 - 1 - 4 - 6 - *
9 - 7 - 9 - 7 - 1 - 9 - *.

Thus the only triangular having all digits 5 is 55.

If $533 \cdots 329$ is to be a perfect square, it must be one of the following three numbers: 9,529 or 5,329. The other possibilities are:

3 - 2 - 3 - *
3 - 2 - 8 - 1 - *
3 - 2 - 8 - 6 - 3 - 4 - *
3 - 2 - 8 - 6 - 3 - 9 - 1 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 2 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 7 - 1 - 4 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 7 - 1 - 9 - *
3 - 2 - 8 - 6 - 3 - 9 - 6 - 7 - 6 - *
3 - 2 - 8 - 6 - 8 - *
3 - 7 - 5 - *
7 - 2 - 4 - *
7 - 2 - 9 - 1 - 3 - 5 - *
7 - 2 - 9 - 1 - 8 - *
7 - 2 - 9 - 6 - *
7 - 7 - 1 - 3 - 1 - *
7 - 7 - 1 - 3 - 6 - 5 - *
7 - 7 - 1 - 8 - *
7 - 7 - 6 - *

Hence the only triangulars having all 6's are 6, 66 and 666.

No number of the form $711 \cdots 105$ is a perfect square because the 10's digit is impossible.

Thus we have eliminated all possibilities and have shown that the above mentioned triangulars are the only ones with repeated digits.

LIST OF REFERENCES

Escott, E. B. 1905. Math. Quest. Educ. Times. 8:33-34.