PYTHAGOREAN TRIANGLES WHOSE LEGS DIFFER BY ONE

Dale C. Koepp and David Ballew
South Dakota School of Mines & Technology
Rapid City, S. D. 57701

INTRODUCTION

A Pythagorean triangle is defined as a perfect right triangle with all integer sides. It is denoted as a triplet (a, b, c) where a and b are legs and c is the hypotenuse of the triangle.

The particular class of Pythagorean triangles which is of interest here is where the two legs, a and b, differ by one. For simplicity, let these triangles be called *unit triangles.* That is, (a, b, c) is a unit triangle when $a^2 + b^2 - c^2 = 0$, $|a - b| = 1$, and a, b, and c are positive integers.

Since this is a problem in number theory, it will be assumed that all of the variables here are positive integers unless otherwise stated.

MAIN THEOREM

In this paper, I wish to prove the following theorem which demonstrates a method of generating all of the unit triangles.

Theorem: The sequence of all unit triangles can be generated as follows:

$$\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{i-1} \\ b_{i-1} \\ c_{i-1} \end{bmatrix}$$

where (a_i, b_i, c_i) is the ith unit triangle and $(3, 4, 5)$ is the first unit triangle in the sequence.

Proof: The recursion formula can be rewritten as follows:

$$\begin{aligned} a_i &= a_{i-1} + 2b_{i-1} + 2c_{i-1} \\ b_i &= 2a_{i-1} + b_{i-1} + 2c_{i-1} \\ c_i &= 2a_{i-1} + 2b_{i-1} + 3c_{i-1}. \end{aligned} \quad (1)$$

If $(a_{i-1}, b_{i-1}, c_{i-1})$ is a unit triangle, it follows that

$$a_i > a_{i-1}, b_i > b_{i-1}, c_i > c_{i-1} \quad (2)$$

Also, using simple algebra,

$$a_i - b_i = b_{i-1} - a_{i-1} \quad (3)$$

and

$$a_i^2 + b_i^2 - c_i^2 = a_{i-1}^2 + b_{i-1}^2 - c_{i-1}^2 \quad (4)$$

From equations (2), (3), and (4) it is obvious that (a_i, b_i, c_i) is also a unit triangle.

We know that $(a_1, b_1, c_1) = (3, 4, 5)$ is a unit triangle, therefore by induction, (a_i, b_i, c_i) is a unit triangle for all i.

Call the set of all unit triangles generated by this process T.

Assumption: There exists a set $T' \neq \phi$ which consists of all the unit triangles not in T.

There must exist a smallest value for a hypotenuse of a triangle in T': Let $(x, y, z) \in T'$ have this smallest value for its hypotenuse.

Then, define (u, v, w) such that

$$\begin{aligned} u &= x + 2y - 2z \\ v &= 2x + y - 2z \\ w &= -2x - 2y + 3z. \end{aligned} \quad (5)$$

Solve for x, y and z

$$\begin{aligned} x &= u + 2v + 2w \\ y &= 2u + v + 2w \\ z &= 2u + 2v + 3w \end{aligned} \quad (6)$$

Because equations (1) and (6) are the same, (u, v, w) is related to (x, y, z) as $(a_{i-1}, b_{i-1}, c_{i-1})$ is related to (a_i, b_i, c_i). This means that if we assume that $u, v, w > 0$, then

$$x > u,\ y > v,\ z > w$$

$$x - y = v - u$$

and

$$x^2 + y^2 - z^2 = u^2 + v^2 - w^2$$

and it follows that (u, v, w) is also a unit triangle. Furthermore, $(u, v, w) \in T$, because if it were

$$(u,v,w) = (a_{i-1}, b_{i-1}, c_{i-1}) \text{ for some } i$$

$$\Rightarrow (x,y,z) = (a_i, b_i, c_i)$$

which cannot be true because $(x, y, z) \in T$. Therefore $(u, v, w) \in T'$. But this is a contradiction because $z > w$ and (x, y, z) was supposed to have the smaller hypotenuse. Therefore, one of u, v, w must be ≤ 0.

If $w \leqq 0$

$$-2x - 2y + 3z \leqq 0$$
$$2x + 2y \geqq 3z$$
$$4x^2 + 8xy + 4y^2 \leqq 9z^2 > 8x^2 + 8y^2$$
$$0 > 4x^2 - 8xy + 4y^2$$
$$0 > (2x - 2y)^2$$

which is impossible.

Without loss of generality we can say that y is larger than x, since x and y are interchangeable.

$$y - x = u - v = 1$$
$$u > v$$

Therefore, we need only consider $v \leqq 0$ because if $u \leqq 0$, then $v < 0$. If $v \leqq 0$,

$$2x + y - 2z \leqq 0$$
$$2x + y \leqq 2z$$
$$4x^2 + 4xy + y^2 \leqq 4z^2 = 4x^2 + 4y^2$$
$$4xy \leqq 3y^2$$
$$4x \leqq 3y$$
$$4x \leqq 3x + 3$$
$$x \leqq 3$$
$$y \leqq 4.$$

From this $(x,y,z) = (3,4,5)$. But $(3,4,5) \in T \Longrightarrow (x,y,z) \notin T'$ which is a contradiction. Therefore, $T' = \phi \Longrightarrow$ all unit triangles are generated by the process defined in the theorem. This finishes the proof of the theorem.

EXAMPLES

The first three unit triangles are:

(3, 4, 5),

(21, 20, 29),

(119, 120, 169).

Using the CDC 3400, I found some later examples to be

(744162497452738090880009564 61,

744162497452738090880009564 60,

105240469650709600546001391989)

(500523684722743250061338526726503,

500523684722743250061338526726504,

707847383223858622658735230185145).

These were generated using the main theorem of this paper.