SYNTHETIC AXIOMS FOR SOME SEGMENTWISE CONNECTED DIFFERENTIAL GEOMETRY SURFACES AND FOR ARCS WHICH MAY SELF INTERSECT

Richard P. Menzel
216B E. Cherry St.
Vermillion SD 57069, USA
Corresponding author email: richardmenzel_7@hotmail.com

ABSTRACT

Moritz Pasch (1882. Vorlesungen uber Neuere Geometrie. Teubner, Leipzig.) synthetically developed Euclidean geometry, using "interval" as a primitive idea. Menzel (1988. Variations on a theme of Euclid. International Journal of Mathematical Education in Science and Technology 19:611-617; 1996. Generalized Synthetic order axioms, which apply to geodesics and other uniquely extensible curves which may cross themselves. Proceedings of the South Dakota Academy of Science 75:11-55.) generalized his work so as to allow the intervals to cross themselves. Examples which satisfy the generalized axioms include differential and metric space geometries and the differential equations of Picard. This paper introduces synthetic axioms which define angles and some segmentwise connected surfaces in classical differential geometry. It also introduces some synthetic axioms which apply to arcs which have two endpoints and which may self intersect.

Keywords
Uniquely extensible segments, maximal-uniquely-extensible-curves, angles, surfaces.

INTRODUCTION

The elements of the universe P are called **points**. **Surfaces** are subsets of the universe and are primitive ideas of this theory in the large. **U segments** (U for uniquely extensible) are primitive subsets of the surfaces. Primitive ideas, such as surfaces, U segments, angles and functions which assign endpoints to U segments and sides to angles receive their meaning from the axioms which they satisfy. One model for surfaces are those in classical three-dimensional differential geometry which are segment-wise connected. The segments contained in these surfaces model the U segments, provided they have two endpoints and are defined on intervals which are less than the period of the defining functions of the segments, if the functions are periodic. This and other examples which may be of interest to the reader are in Example 2.3 of (Menzel 1996), which is quoted in the Appendix 1 of this paper. Our models satisfy our axioms but more axioms would probably be needed to uniquely characterize these geometries. We can show that for every point A in every surface S, there are rays (collectively called **Rays(S, A)**), each with initial point A, whose elements are contained in S. **Angles(S, A)** is a subset of the powerset of Rays(S, A), with primitive elements called **angles**. Intuitively an angle is a stretch of rays with common initial point; we define it axiomatically by (mostly) replacing, in the theory of U segments, the word "point" with "ray", "U segment" with "angle", and "endpoint of a U segment" with "side of an angle". Uniquely (radially) extensible angles are used to axiomatize U-segment-wise-connected surfaces. Angles may cross themselves and be arbitrarily large.

The previous papers might encounter difficulties: e.g., with a model which includes the surface in R^3 which is $\{(x, y, 0) : y \geq 0\} \cup \{(x, y, y^2) : y \leq 0\}$ and also includes the tangent plane $z = 0$. The geodesic segment $\{[0, y, 0] : 0 \leq y \leq 1\}$ is not uniquely extensible in this model: it can extend in the parabolic cylinder and in the plane. In these previous papers there was only one surface, which equaled the universe. We have augmented some of the axioms of (Menzel 1996) so that they are satisfied in multiple surfaces. An augmented axiom is indicated with a "+".

A list of notations follows the references.

AXIOMS

Universe P is a nonempty set with elements called **points**. Ξ is a nonempty subset of the powerset of P with elements called **surfaces**. $\sum$ is a nonempty subset of the powerset of P with elements called **U segments** such that every U segment is a subset of a surface and $\forall S \in \Xi$, $\exists$ a $\lambda \in \sum$ such that $\lambda \subseteq S$.

Axiom 1. $\forall \sigma \in \sum$, $|\sigma| > 2$. (|A| means the number of elements in set A.)

Definition of endpoints and notation for U segments. Let F map $\sum$ into $\binom{P}{2}$ (which is the set of all subsets of P which have exactly two elements) such that, $\forall \sigma \in \sum$, $F(\sigma) \subseteq \sigma$. If $F(\sigma) = \{A, B\}$, then A and B are called endpoints of σ, and σ may be written **[AB]** or **[BA].**

Definitions (from Menzel 1988, 1996 and Appendix of 2002). **U segments λ, η are called A related** (intuitively, they have the same direction from A) if $A \in F(\lambda) \cap F(\eta)$, if $\lambda \subseteq \eta$ or $\eta \subseteq \lambda$ and if $\forall \sigma \in \sum$, such that $\lambda \subseteq \sigma \subseteq \eta$ or $\eta \subseteq \sigma \subseteq \lambda$, $A \in F(\sigma)$. **We may write λ A η. If also $\lambda \subseteq \eta$, we may write λ A$\subseteq$ η.**

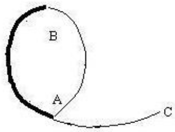


Figure 1

See the heuristic example in Figure 1. To generate curve ABAC move counterclockwise (rotate against the direction of a clock hand) from A to B to A and then directly to C. It has endpoints A, C and a tangent line at interior point A. Let curve ABAC be a U segment as well as any subcurve of it, with two endpoints, which has a tangent line (possibly vertical) at each interior point. Heavily inked curve BA is not A related to curve ABAC because, e.g., the simple curve BC is a U segment which contains it, is contained in ABAC and does not have endpoint A. The curve ABAC is A related to lightly inked curve AB. For a more specific example, Figure 1 graphs, in polar coordinates, $r(\theta) = 2 + 4\sin\theta$ on $[7\pi/6, 2\pi]$ (the domain of ABAC). Point A = $(0, 7\pi/6) = (0, 11\pi/6)$, B = $(-2, 3\pi/2)$ and C = $(2, 2\pi)$. Restrict the domain to a closed subinterval (other than $[7\pi/6, 11\pi]$) to get other U segments. Thus "heavily inked curve BA" is defined on $[3\pi/2, 11\pi/6]$, "simple curve AC" on $[11\pi/6, 2\pi]$, "simple curve BC" on $[3\pi/2, 2\pi]$ and "lightly inked curve AB" on $[7\pi/6, 3\pi/2]$.

Theorem 1.1 (in Menzel 1988, 1996 and 2002). [AB] A [AB] and [AB] B [AB]. If λ A η, then η A λ.

Axioms (perhaps generalized) are numbered as in (Menzel 1988,1996 and 2002).

Axiom 2. $\forall$ [AB] $\in \sum$, $\forall$ P $\in$ [AB] – {A, B}, $\exists$ [AP] $\in \sum$ such that [AP] A$\subseteq$ [AB].

A generalization of Axiom 3 in (Menzel 1988, 1996 and 2002) will be proved in Theorem 4, so **this paper has no Axiom 3.**

Axiom 4+. $\forall$ S $\in \Xi$, $\forall$ [AB] $\in \sum$ and $\subseteq$ S, $\exists$ a point X $\neq$ B and an [AX] $\in \sum$ and $\subseteq$ S, $\ni$ [AB] A$\subseteq$ [AX].

If one removes references to S and that X $\neq$ B, Axiom 4+ becomes **Axiom 4 of the Menzel references.**

Axiom 5+ will follow Axiom 8+.

Definition. $\forall$ [AB], [AC] $\in \sum$ we say **[AB] and [AC] are opposite (at common endpoint A) or [AB] oppA [AC]** if $\exists$ [BC] $\in \sum$ such that [AB] $\cup$ [AC] = [BC] and $\forall$ $\sigma \in \sum$, $\sigma \subseteq$ [AB] $\cap$ [AC] is false. We may also say **[BA] opp [AC] in [BC].**

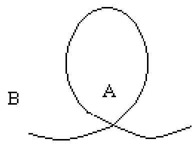


Figure 2

See the heuristic example in figure 2. To generate curve BAC go from B to A, then counterclockwise to A, then directly to C. Let curve BAC (without corner) be a U segment as well as any subcurve which has two endpoints and a tangent line (possibly vertical) at each interior point. The union of simple curves BA and AC is not a U segment because it does not have a tangent line at A. To generate [BA]* go from B to A, then counterclockwise from A to A. To generate [AC]* go counterclockwise from A to A and then direct to C. [BA]* and [AC]* are not opposite since their intersection contains a U segment. For the same reason [BA]* is not opposite simple curve [BA]. [AC]* and (simple curve) BA are opposite since their intersection $\not\supseteq$ a U segment and their union is a U segment (the curve BAC). Figure 2 (above) graphs $r = 2 + 4\sin\theta$ on $[\pi, 2\pi]$ (the domain of BAC). A = $(0, 7\pi/6) = (0, 11\pi/6)$, B = $(2, \pi)$ and C = $(2, 2\pi)$. Restrict the domain to a closed subinterval (other than $[7\pi/6, 11\pi/6]$) to get other U segments. Simple curve [BA] is defined on $[\pi, 7\pi/6]$, simple curve [AC] on $[11\pi/6, 2\pi]$, [BA]* on $[\pi, 11\pi/6]$ and [AC]* on $[7\pi/6, 2\pi]$.

Axiom 6 (Subtraction of U segments). ∀ A, B, C ∈ P, ∀ [BA], [BC] ∈ ∑ such that [BA] B⊆ [BC] and C ≠ A, ∃ a unique [AC] ∈ ∑ such that [BA] oppA [AC] in [BC]. (Theorem 7 proves the uniqueness of [AC].)

Theorem 1. ∀ S ∈ Ξ, ∀ [AB] ∈ ∑ and ⊆ S, ∃ [BC] ∈ ∑ and ⊆ S such that [AB] oppB [BC].

Prove with Axioms 4+ and 6.

Theorem 2. ∀ [AB] ∈ ∑, |[AB]| > 4. (|n| means the number of elements in set n. Later we prove |[AB]| ≥ c_0.)

Proof. ∃ C ∈ [AB] – {A, B}. By Axiom 2, ∃ [AC] ∈ ∑ such that [AC] A⊆ [AB]. [AC] ≠ [AB], as F([AC]) ∈ $(^{P}{}_{2})$. So [AC] A⊆ [AB], proper. Since |[AC]| > 2, |[AB]|> 3. Suppose |[AB]| = 4, so |[AC]| = 3. Let D ∈ [AB] – {A, B, C}. If B ∈ [AC], then D ∉ [AC] and ∃ [AB]* A⊆ [AC], proper, such that |[AB]*| = 2. Contradiction, and a similar contradiction if D ∈ [AC]. This contradicts |[AB]| = 4.

Axiom 6'. For all points A, C, ∀ λ, η, [AC] ∈ ∑ such that λ ∪ η = [AC], either C ∈ F(λ) and A ∈ F(η), or C ∈ F(η) and A ∈ F(λ), or λ ⊆ η or η ⊆ λ.

Axiom 7 (Addition of U segments). ∀ point A, ∀ [AB], [AC], [AB'], [AC'] ∈ ∑, if [AB] oppA [AC], [AB'] A [AB], [AC'] A [AC], B' ≠ C' and, ∀ σ ∈ ∑, σ ⊆ [AC'] ∩ [AB'] is false, then ∃ [B'C'] ∈ ∑ such that [AC'] opp [AB'] in [B'C'].

Theorem 3. ∀ S ∈ Ξ, ∀ α, β ∈ ∑ (with α, β ⊆ S), if α A⊆ β, then ∃ β'' ∈ ∑ and ⊆ S such that α oppA β'' and β oppA β''. (I.e., A related U segments ⊆ S have a common opposite at A which is ⊆ S.)

Proof. Using Theorem 1, let β' be oppA β, with β' ⊆ S. See proof in our Appendix 2.

Axiom 8+. ∀ S ∈ Ξ, ∀ γ, η, κ ∈ ∑ (with γ, η, κ ⊆ S), if γ oppA κ and η oppA κ, then γ A η.

Removing references to surface S gives A**xiom 8 of (Menzel 1996 and 2002).**

Theorem 4. ∀ S ∈ Ξ, ∀ λ, η, κ ∈ ∑ (with λ, η, κ ⊆ S), if λ A η and η A κ then λ A κ. (Deleting references to S gives **Axiom 3 of (Menzel 1988, 1996 and 2002)).**

Proof. Using Theorem 3, let ϱ ∈ ∑ and be ⊆ S such that ϱ oppA λ and ϱ oppA η and let φ ∈ ∑ and be ⊆ S such that φ oppA η and φ oppA κ. Using Axiom 8+ and ϱ oppA η and φ oppA η we get ϱ A φ, so either φ ⊆ ϱ or φ ⊇ ϱ. Let X ∈ (ϱ ∩ φ) such that X ≠ any endpoint of λ, η or κ (see Theorem 2). Let [AX] be A⊆ (ϱ ∩ φ) (which = ϱ or φ). Using Axiom 7, [AX] is opposite η. Since η A κ, [AX] oppA η, X ∉ F(κ) and [AX] ∩ κ ⊉ a σ ∈ ∑ (remembering that [AX] A⊆ φ and φ oppA κ), Axiom 7 says [AX] oppA κ. Substituting λ for κ in the preceding sentence, [AX] oppA λ. Since [AX] oppA both λ and κ (and [AX], λ and κ ⊆ S), so λ A κ, by Axiom 8+).

Theorem 5. If [BA] opp [AC] in [BC] and [BA] opp [AC]' in [BC], then [AC] = [AC]'. Apply this to Axiom 6.

Proof. [AC] A [AC]', by Axiom 8+. Also [AC] C [AC]' because, by Theorem 1.9 of (Menzel 1996) which is quoted in this paper's Appendix 2, [AC] and [AC]' are both C related to [BC] and so by Theorem 4 they are C related to each other. Therefore [AC] = [AC]', by Theorem 1.2 of (Menzel 1988), quoted in this paper's Appendix 2. Or see Theorem 1.14 in Appendix 2, quoted from (Menzel 1996).

Definition of ray. Let [AB] ∈ ∑ and be ⊆ S. Theorems 1.1 and 4 say that {η ∈ ∑ : η A [AB] and ⊆ S} is an equivalence class. We call it **Ray([AB], S)>**. Point A is called the **i*nitial point of the ray*:** put it first in ([AB], S)>. **[AB]**> is the notation for the ray in (Menzel 1988, 1996 and 2002).

Definition. If λ, κ ∈ ∑ and λ A⊆ κ, proper, we say that **λ < κ.**

Theorem 6. A ray is a linearly ordered set. (This means that for all α, β in the ray, exactly one of the relations α < β, α = β or β < α holds, the transitive law holds (see Theorem 4) and no U segment is less than itself.)

Definition of a maximal-uniquely-extensible-curve (called muec). ∀ S ∈ Ξ, ∀ [AB] ∈ ∑ and ⊆ S, the Union{[XY] ∈ ∑ and ⊆ S: [XY] A [AB] or [XY] B [AB]} is called a **maximal-uniquely-extensible-curve or muec and may be written <[AB], S>.** The set of all muecs is called **Λ**.

Axiom 5+. ∀ S ∈ Ξ, ∀ A, B ∈ S, ∀ Γ ∈ Λ and ⊆ S, ∀ [AB] ∈ ∑ and ⊆ Γ, ∀ X ∈ Γ, either ∃ [AX] ∈ ∑ and ⊆ Γ such that [AB] A [AX] or ∃ [BX] ∈ ∑ and ⊆ Γ such that [AB] B [BX].

Axiom 5 of (Menzel 1988,1996 and 2002) said $\forall\ \lambda,\ \eta \in \sum$, if $\lambda \subseteq$ the union of the set of all U segments which are related to η (at either endpoint of η), which union is called $<\eta>$, then $<\lambda> = <\eta>$. A generalization of this axiom is proved in Theorem 7 (b), which allows multiple surfaces.

Theorem 7 (trivial). $\forall S \in \Xi, \forall A, B \in S, \forall \Gamma \in \Lambda$ and $\subseteq S, \forall [AB] \in \sum$ and $\subseteq \Gamma$:

(a) Union$\{\sigma \in \sum$ and $\subseteq \Gamma$: either σ A [AB] or σ B [AB]$\} = \Gamma$. (Prove with Axiom 5+.)

(b) If this union (called <[AB], S>) contains $[VW] \in \sum$, then it equals <[VW>, S>. (By part (a), both sets $= \Gamma$.)

Theorem 8. $\forall [AB] \in \sum, |[AB]| \geq c_0$. ($|n|$ means the number of elements in set n.)

Proof. Suppose $|[AB]| = n$. $\forall X \in [AB] - \{A, B\}$ take one [AX] A$\subseteq$ [AB]. Name these X's $A_1, A_2, \ldots,$ A_{n-2} so that, by Theorem 4, $[AA_1]$ A$\subseteq$ $[AA_2]$ A$\subseteq, \ldots, \subseteq [AA_{n-2}]$ A$\subseteq$ [AB] (all proper $\subseteq$'s because, $\forall \sigma \in \sum, F(\sigma) \in ({}^{P}{}_{2})$). $|[AA_1]| > 2$, so $|[AA_{n-2}]| > n-1$, so $|[AB]| > n$. Contradiction.

Axiom 9. $\forall S \in \Xi, \forall \{A, B\} \in ({}^{S}{}_{2}), \exists [AB] \in \sum$ and $\subseteq S$. (Surfaces are U segmentwise connected.)

Theorem 9. $\forall \Gamma \in \Lambda, \forall \{A, B\} \in ({}^{\Gamma}{}_{2}), \exists \sigma \in \sum$ such that $\sigma \subseteq \Gamma$ and $F(\sigma) = \{A, B\}$.

Proof. By definition of a muec, $\exists \lambda \in \sum$ such that $\lambda \subseteq \Gamma$. Let $F(\lambda) = \{C, D\}$. By Axiom 5+, either $\exists [CA] \subseteq \Gamma$ such that [CA] C λ or $\exists [DA] \subseteq \Gamma$ such that [DA] D λ. In the first case $<\lambda, \Gamma> = <[CA], \Gamma>$, by Theorem 7, and either $\exists [AB] \subseteq \Gamma$ such that [AB] A [AC] (q.e.d.) or $\exists [CB] \subseteq \Gamma$ such that [CB] C [CA] (use Axiom 6). In the second case, the proof is similar.

Axiom 10. $\forall S \in \Xi, \exists A, B, C \in S$ such that, $\forall \Gamma \in \Lambda, \{A, B, C\} \subseteq \Gamma \subseteq S$ is false.

ANGLES

$\forall S \in \Xi, \forall A \in S, \exists$ the nonempty set **Rays(S, A)** of all rays, with initial point A, whose (U segment) elements are $\subseteq$ S. **Angles(S, A)** is a nonempty subset of the powerset of Rays(S, A) with (primitive) elements called **angles.**

Angle Axiom 1. $\forall$ angle $\alpha \in$ Angles(S, A), $|\alpha| > 2$.

Notation and definition of sides of an angle. Let G map every Angles(S, A) into $({}^{\mathrm{Rays(S,\,A)}}{}_{2})$ such that, $\forall \alpha \in$ Angles(S, A), $G(\alpha) \subseteq \alpha$. If $G(\alpha) = \{$[AB]>, [AC]>$\}$, then [AB]>, [AC]> are called **sides of α and we write α as ([AB]>, [AC]>) or ([AB]>, [AC]>, S).**

Angle Axioms 1, 2, 6, 6' and 7 are analogous to Axioms 2, 6, 6' and 7 by replacing "point" with "ray", "U segment" with "angle" and "endpoint" with "side". If the universe is a single surface (as was the case in some models of (Menzel 1996), quoted in Appendix 1), then Angle Axioms 4 and 8 are also analogous to Axioms 4+ and 8+. If the universe is not a single surface, then Angle Axioms 4 and 8 are not analogous to Axioms 4+ and 8+. Angle Theorem 7 part (b) will be analogous to Axiom 5 of (Menzel

1988, 1996 and 2002), if references to S are deleted. Theorems and proofs about angles which follow from analogous axioms are also analogous to those concerning U segments. The same replacements give definitions of related and opposite angles and Eudoxus-Dedekind continuity of angles. These substitutions follow. The best result is Angle Theorem 7, the only theorem in this paper which uses Axiom 9.

Definition. Angles α, β are [AB]> related (intuitively, they have the same direction of rotation from common side [AB]>) if [AB]> ∈ G(α) ∩ G(β) (that is, ray [AB]> is a common side of α and β), if α ⊆ β or β ⊆ α and if ∀ angle λ such that α ⊆ λ ⊆ β or β ⊆ λ ⊆ α, then [AB]> ∈ G(λ). Write **α [AB]> β** . If also α ⊆ β, write **α [AB]>⊆ β**.

Angle Theorem 1.1. (a) ∀ S ∈ Ξ, ∀ A, B, C ∈ S, every Angle ([AB]>,[AC]>,S) is both [AB]> and [AC]> related to itself (reflexive). **(b)** ∀ S ∈ Ξ, ∀ A, B, C, D ∈ S, ∀ Angles ([AB]>,[AC]>,S) and ([AB]>,[AD]>,S), if ([AB]>,[AC]>,S) [AB]> ([AB]>,[AD]>,S), then ([AB]>,[AD]>,S) [AB]> ([AB]>,[AC]>,S) (symmetric).

Angle Axiom 2. ∀ angle ([XY]>,[XZ]>,S) ∀ ray [XW]> ∈ [([XY]>,[XZ]>,S) – {[XY]>,[XZ]>}], ∃ an angle ([XY]>,[XW]>,S) which is [XY]>⊆ ([XY]>,[XZ]>,S).

Angle Axiom 4. ∀ angle ([AB]>, [AC]>, S) ∃ angle ([AB]>, [AD]>, S) such that [AD]> ≠ [AC]> and ([AB]>, [AC]>, S) [AB]>⊆ ([AB]>, [AD]>, S), proper.

Definition. ([AB]>, [AC]>) = α is opposite ([AB]>, [AD]>) = β (at common side [AB]>) if α ∪ β = an angle ([AC]>, [AD]>) and, ∀ angle λ, λ ⊆ α ∩ β is false. We may also write **α opp[AB]> β** or write **α opp β in ([AC]>, [AD]>).**

Angle Axiom 6 (Subtraction of angles). ∀ angles α, ϱ such that α [XY]>⊆ ϱ and G(α) ≠ G(ϱ), ∃ a unique angle β such that β opp α in ϱ (and G(β) = ((G(α) ∪ G(ϱ)) – {[XY]>}).

Angle Theorem 1. ∀ Rays(S, A), ∀ angle ([AB]>, [AC]>, S) ⊆ that Rays(S, A), if [AX]> ∈ G(([AB]>, [AC]>, S)), then ∃ angle β ⊆ that Rays(S, A) such that [AX]> ∈ G(β) and β opp[AX]> ([AB]>, [AC]>, S).

Angle Theorem 2. ∀ angle α, |α| > 4. (Proof is analogous to the proof of theorem 2.)

Angle Axiom 6'. If ϱ, α and β are angles, G(ϱ) = {η_1>, η_2>} and ϱ = α ∪ β, then either (1) η_1> ∈ G(α) and η_2> ∈ G(β) or (2) η_1> ∈ G(β) and η_2 ∈ G(α) or (3) α ⊆ β or (4) β ⊆ α.

Angle Axiom 7 (Addition of angles). ∀ points A, B, ∀ ray [AB]>, ∀ angles α, β, ϱ, κ, if α opp[AB]> β, ϱ [AB]> α, κ [AB]> β, G(ϱ) ≠ G(κ) and, ∀ angle λ, λ ⊆ (ϱ ∩ κ) is false, then ϱ ∪ κ is an angle and G(ϱ ∪ κ) = ((G(ϱ) ∪ G(κ)) – {[AB]>}).

Angle Theorem 3. ∀ Rays(S, A), ∀ angles α, β ⊆ Angles(S, A), if α [AB]>⊆ β, then ∃ β'' ∈ Angles(S, A) such that α opp[AB]> β'' and β opp[AB]> β''.

Angle Axiom 8. ∀ Rays(S, A), ∀ α, β, ϱ ⊆ that Rays(S, A), if α opp[AB]> ϱ and β opp[AB]> ϱ, then α [AB]≥ β.

Angle Theorem 4. ∀ Angles(S, A), ∀ α , β, γ ∈ Angles(S, A), if α [AB]≥ β and β [AB]≥ γ, then α [AB]≥ γ. (See proof of Theorem 4.)

Angle Theorem 5. ∀ S ∈ Ξ, ∀ A, B, C, D ∈ S, ∀ Angles ([AB]>,[AC]>,S), ([AC]>,[AD]>,S), ([AC]>,[AD]>,S)* and ([AB]>,[AD]>,S), if ([AB]>,[AC]>,S) is opposite both ([AC]>,[AD]>,S) and ([AC]>,[AD]>,S)* in ([AB]>,[AD]>,S), then ([AC]>,[AD]>,S) = ([AC]>,[AD]>,S)*.

Definition. If α [AB]≥⊆ β, proper, we may say **α < β.**

Angle Theorem 6. If ([AB]>, [AC]>, S) ∈ Angles(S, A), then {α ∈ Angles(S, A) : α [AB]≥ ([AB]>, [AC]>, S)} is linearly ordered.

Angle Axiom 5. ∀ S ∈ Ξ, ∀ A ∈ S, ∀ ([AX]>, [AY]>,S) (called β) ∈ Angles(S, A) and ∀ [AZ]> ∈ Rays(S, A), ∃ α ∈ Angles(S, A) such that [AZ]> ∈ G(α) and either α [AX]≥ β or α [AY]≥ β.

Angle Theorem 7. ∀ S ∈ Ξ, ∀ A ∈ S, ∀ ([AX]>, [AY]>, S) (called β) ∈ Angles(S, A):

(a) Union{α ∈ Angles(S, A) : either α [AX]≥ β or α [AY]≥ β} = Rays(S, A) (prove with Angle Axiom 5).

(b) If the Union in part (a) contains λ = ([AZ]>, [AW]>, S), then it = Union{ϱ ∈ Angles(S, A) : ϱ [AZ]≥ λ or ϱ [AW]≥ λ} (by part (a), they both equal Rays(S, A)).

(c) Union{σ ∈ ∑ : σ ∈ Union{Rays(S, A)} = S (prove with Axiom 9).

(d) S = Union{σ ∈ ∑ : σ ∈ Union{all rays ∈ Union{α ∈ Angles(S, A) : α [AX]≥ β or α [AY]≥ β}} (prove with parts (a) and (c)).

(e) Parts a-d do not depend on the vertex of the angle. E.g., replacing β with η = ([LM]>, [LN]>, S) ∈ Angles(S, L) in part (d) gives S = Union{σ ∈ ∑ : σ ∈ Union{all rays ∈ Union{α ∈ Angles(S, L) : α [LM]≥ η or α [LN]≥ η}.

Angle Theorem 8. ∀ angle α, |α| ≥ c_0. (See proof of Theorem 8.)

Angle Theorem 9. ∀ S ∈ Ξ, ∀ A ∈ S, ∀ Rays(S, A), ∀ {[AX]>, [AY]>} ∈ $\binom{\text{Rays(S, A)}}{2}$, ∃ angle α ⊆ Rays(S, A) such that G(α) = {[AX]>, [AY]>}. (See proof of Theorem 9.)

SOME SYNTHETIC AXIOMS FOR ARCS WITH TWO ENDPOINTS, WHICH MAY SELF INTERSECT

An arc is frequently defined as a continuous image in R^n of a closed real number interval. Thus, an arc is a set of points. In axiomatizing arcs with two endpoints we will also produce some arc-orders, but not those which repeat subarcs (such as periodic images of the closed line interval). We do not discuss arcs with one endpoint. Perhaps coordinates could be introduced by generalizing (Menzel 2002).

There is a nonempty, primitive subset of the powerset of universe P called "**Arcs**" such that $\forall$ arc $\sigma \in$ Arcs, $|\sigma| > 2$. Let H map Arcs into $(^{P}{}_{2})$ (which is the set of all subsets of P which have exactly two elements) such that $\forall$ arc σ, $H(\sigma) \subseteq \sigma$. If $H(\sigma) = \{A, B\}$, then A, B are called endpoints of σ, and σ may be written /**AB**/. Every arc /AB/ contains the elements of a (primitive) set of arcs with endpoints A and may contain the elements of more than one such (primitive) set. These primitive sets are called "**arc-orders of /AB/**". An arc-order of /AB/ may be written **//AB//.**

Example 1. Think of the set of arcs $\{[0, x] : 0 < x \leq 1\}$ as an ordering of the arc [0, 1].

Example 2. A U segment [AB] is an arc. {U segments [AX] : [AX] A$\subseteq$ [AB]} is an ordering of [AB], as is {U segment [BY] : [BY] B $\subseteq$ [AB]}.

Example 3. Refer to Figure 2. You may order arc BAC by having the arcs traverse the loop from (A to A) either clockwise or counter-clockwise, (arcs may have corners), giving several arc-orders of BAC.

Arcs Axiom 1. Every arc is an element of each of its arc orders.

Arcs Axiom 2. $\forall$ A, B belonging to universe P, $\forall$ arc /AB/, $\forall$ arc-order //AB// of this /AB/, $\forall$ point X belonging to /AB/ - {A,B}, $\exists$ an arc /AX/ belonging to //AB//. Its arc-order //AX// is a subset of this //AB//.

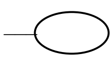
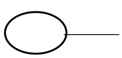
Arcs Axiom 3. If /AC/ and /AD/ belong to the same arc-order of /AB/, then one contains the other. If arc-orders //AC// of /AC/ and //AD// of /AD/ are contained in the same arc-order of /AB/, then either //AB// $\subseteq$ //AD// or //AD// $\subseteq$ //AB//.

Definition. Arcs /AB/, /BC/ are said to be **arc-opposite** (at B, their common end-point) if their intersection does not contain an arc and A $\neq$ C.

Arcs Axiom 7 (Addition of arcs). If the intersection of arc /AB/ and arc /BC/ does not contain an arc and A $\neq$ C, then $\exists$ an arc /AC/ such that /AC/ = (/AB/ $\cup$ /BC/), that is, /AB/ is arc-opposite /BC/ in /AC/.

Arcs Axiom 6 (Subtraction of arcs). For all points A,B, $\forall$ /AB/, $\forall$ arc-order //AB// of this /AB/, $\forall$ X belonging to /AB/ - {A,B}, $\forall$ arc /AX/ belonging to this arc-order of this /AB/, $\exists$ a unique arc /BX/ which is arc-opposite to this /AX/ in this /AB/.

Definition of continuity of arcs. Suppose $\forall$ A oriented set [α} which is contained in arc-order //AB// either (a) $\exists$ an arc /AX/ $\in$ //AB// with arc-order //AX// $\subseteq$ //AB//, such that [α} = //AX//- {/AX/} (X may equal B) or (b) $\exists$ an arc /BA/* which contains all of the opposites (in /AB/) of elements of //AB// - [α} and is contained in every opposite (in /AB/) of elements of [α}. We then say that arc **//AB// is continuous.**

Part (a) or  Part (b)  **Figure 3 (Examples)**

A X B A X=B A B

U segments may traverse cycles in either direction. In part (b), [α} is a subset of the loop.

This paper added the primitive ideas “Surfaces” and “Angles” to those of the preceding papers, as well as adding axioms about them, to give synthetic definitions of angles and segment-connected surfaces which are applicable to classical differential geometry. Many axioms about angles may be derived from axioms about U segments by replacing the words “angle” for “U segment”, “ray” for “point” and “side” for “endpoint”. Additional axioms (hopefully in the large) should be added, for all such surfaces or for subsets of this class of surfaces which are of interest to the reader. Why not construct surfaces from other solution curves of the Picard theorem, rather than from geodesics?

Some synthetic axioms for arcs with two endpoints, which may self-intersect, were also discussed. Perhaps coordinate systems for some of these arcs could be introduced by generalizing (Menzel 2002).

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APPENDIX 1

Example 2.3. This is quoted from (Menzel 1996 and 2002). Let set universe S be R^n or a surface in classical differential geometry or an n-dimensional manifold ($n > 1$) which contains a collection C of differentiable curves defined on ($-\infty$, ∞), whose first derivatives never equal the null vector, and satisfying "through every point in S and for every direction on S there is a unique curve in C with that direction". (If parameter t may be chosen so that tangent vectors g'(t) always have length 1, then a curve need be defined only on any open real number interval.) Let the U segments be subcurves, defined on intervals [a, b], provided that (endpoint) $g(a) \neq g(b)$ and $b - a$ is less than the period of g(t), if g(t) is periodic. An example of a system of such curves is what (Laugwitz 1960 and 1965), pages 190-197, calls a "space of paths", the solutions of a system of equations $x''^i + 2G^i(x; x') = 0$ and $G^i(x; Ax') = A^2 G^i(x; x')$ (where x is a point in a manifold and ' means differentiation). This includes geodesics in Finsler Spaces, auto-parallels in spaces with affine connection (and in Riemannian Spaces) and geodesics in differential geometry. Another example is a system of solution curves ($y_1(t)$, $y_2(t)$, …, $y_n(t)$) of second order differential equations $y_i'' = F_i(t, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')$ (where i = 1, 2, ,…, n; two endpoints: domain lengths less than a period, if curves are periodic) whose first derivatives never equal the null vector and where F_i is continuous in some neighborhood N of (t_0, $y_1(t_0)$, $y_2(t_0)$, …, $y_n(t_0)$, $y_1'(t_0)$, $y_2'(t_0)$ …, $y_n'(t_0)$) and satisfies Lipschitz conditions in all arguments except the first on N (Picard's Theorem). Theorem 5.1 of (Menzel 1996) proves that Example 2.3 satisfies Axioms 1 through 8 and that every U segment is continuous. (Continuity is defined in Appendix 3 of this paper.)

In this paper we have modified a few axioms, so we will modify some proofs from the earlier papers. We will restrict our model to three-dimensional segmentwise connected differential geometry, with geodesics as models for muecs and the U segments are subcurves, of the geodesics, defined on intervals [a, b], provided that (endpoint) $g(a) \neq g(b)$ and $b - a$ is less than the period of g(t), if g(t) is periodic. Other models are suggested by Example 2.3. In these proofs, the image of interval [XY] is written g[x,y]. In (Menzel 1988 and 1996) a U segment [XY] was written as a Greek letter with subscripts A, B. Inclusion of the condition "$\forall$ surface S," in some of the new axioms is nothing new since, in just quoted Example 2.3 of (Menzel 1996), we considered universe S to be a single surface.

Lemma 2 from Theorem 5.1 of (Menzel 1996). (Part 1) If U segments α and β satisfy $\alpha = f[a,b]$, $\beta = f[a,c]$ and $b < c$, then α and β are f(a) related. (Part 2) If instead $\alpha = f[b,a]$, $\beta = f[c,a]$ and $c < b < a$, then α and β are f(a) related.

To prove this model satisfies Axiom 4+, let [AB] = f [a, b], with A = f(a) and B = f(b). Choose $c > b$ such that $f(c) \neq f(a)$ (and $c < a + p$, if f(t) has period p) and [f(a) f(c)] is $\subseteq$ S. (Such a c exists. Otherwise f(t) = f(a) for all t in some interval [b, x] such that $x < a + p$ and limit f(t) (as $t \to b$ from the right) is f(b) =

f(a), so that A = B and [AB] has only one endpoint. No.) Let [AC] = [f(a) f(c)] = f[a, c], which is f(a) related to f[a, b] by Lemma 2. If f(c) $\neq$ f(b) = B, let X = f(c), and [f(a) f(c)] = [AX] is the U segment we seek (i.e., [AB] A$\subseteq$ [AX], proper, and X $\neq$ B). If f(c) = f(b), we show $\exists$ d such that b < d < c, f(d) $\neq$ f(a) = A and f(d) $\neq$ f(b) = B. As just argued, there is no interval [bx], with b < x < c, whose elements all satisfy f(t) = A. Choose a sequence t_n in interval (bc) such that f(t_n) $\neq$ f(a) and limit t_n = b (as $t_n \rightarrow$ b from the right). If every element of this sequence satisfies f(t_n) = f(b), then we have a sequence of U segments [f(a), f(t_n)] for which limit [f(t_n) - f(b)] / [t_n – b]] (as $t_n \rightarrow$ b from the right) = the null vector. But derivatives of curves in our model never equal the null vector. Therefore $\exists$ an element d of this sequence such that f(d) $\neq$ f(b) and such that b < d< c and f(d) $\neq$ f(a) =A and f(d) $\neq$ f(b) = B. Let X = f(d) and [AX] = [f(a) f(e)] is A$\subseteq$ [AX], proper.

Proof that Example 2.3, stated above, satisfies Axiom 5+. Let Γ = g(real number line) and let [AB] = g[a,b], with g(a) = A and g(b) = B. If X $\in \Gamma$ and X = g(x), then either x < a or x = a or a < x < b or x = b or x > b. **If x < a**, let [BX] = g[x,b], which is g(b) =B related to g[ab] = [AB] by Lemma 2**. If x = a,** then [BX] = g[a,b] = [AB], so [BX] B [BA] by Theorem 1.1. **If a < x < b,** let [AX] = g[a,x]. Since [ax] $\subseteq$ [ab], [AX] A [AB] by Lemma 2. (Similarly g[x,b] is B related to [AB] = g[a,b], by Lemma 2.) **If x = b,** let [AX] = g[a,b] = [AB], so [AX] A [AB]. **If b < x,** then [AX] = g[a,x], which is A related to [AB] = g[a,b] by Lemma 2.

Lemma 1. If U segment α = f[a,b] is $\subseteq$ U segment β = g[c,d], then (1) there are t* and t** in [c,d] and constant $\lambda \neq 0$ such that g(t) = f(λt - λt* + a) = f(λt - λt** + b), with t* < d if λ > 0 and t** < d if λ < 0. (Special case: If [a,b] and [c,d] are disjoint and g(t) = f(t), then g(t) has a period $\leq$ |t* - a| < the maximum distance between two points in [a,b] $\cup$ [c,d].) Also (2): f[a,b] is the image under g(t) of a unique interval in [c,d]: it is [t*,t* + (b - a)/λ] = [t** - (b - a)/λ,t**] if λ > 0 and it is [t**,t** - (b - a)/λ] = [t* + (b - a)/λ, t*] if λ < 0. Thus if f[a,b] = g[x,y] and c $\leq$ x < y $\leq$ d, then t* = x and t** = y if λ > 0 and t* = y and t** = x if λ < 0.

Lemma 6. If α is opposite β (satisfying $\gamma = \alpha \cup \beta$ = g[a,c]), then there is a unique b between a and c such that α = g[a,b] and β = g[b,c] or vice versa.

Proof that Axiom 8+ is satisfied in example 2.3. We restate the axiom, substituting B for A.

Axiom 8+. $\forall$ S $\in \Xi$, $\forall$ $\gamma, \eta, \kappa \in \sum$ and $\subseteq$ S, if γ oppA κ and η oppA κ, then γ A η.

Let κ = [AB], γ = [AC] and η = [AD]. Let [AB] be opposite [BC] in [AC] = g[a,c] and let [AB] be opposite [BD] in [AD] = f[a',d]. Lemma 6 says there is a unique b between a and c such that [AB] = g[a,b] and [BC] = g[b,c] or vice versa. Similarly there is a unique b' between a' and d such that [AB] = f[a',b'] and [BD] = f[b',d] or vice versa. (**Case 1**) Let [AB] = g[a,b] = f[a',b'], so that [BD] = f[b',d] and [BC] = g[b,c]. Use Lemma 1 on f[a', b'] = [AB] $\subseteq$ g[a, c]. If **λ > 0**, g(t) = f(λt - λa + a'), noting that t* =

a by the last sentence of Lemma 1. Therefore f(t) = g(t/λ + a - a'/λ) and [AB] = f[a',b'] = g[a,a + (b' - a')/λ] = g[a,b], so that b = a + (b' - a')/λ. Then [BD] = f[b',d] = g[a + (b' - a')/λ,a + (d - a')/λ] = g[b, a + (d - a')/λ]. [BC] = g[b,c] is g(b) related to [BD] by Lemma 2. If **λ < 0**, g(t) = f(λt - λb + a'), f(t) = g(t/λ + b - a'/λ) and [AD] = f[a',d] = g[b + (d - a')/λ, b]. Therefore [AD] has endpoint g(b) = B (in common with [AB] and [BC] , which have only one common endpoint g(b) = B, because of [AC]). Therefore either A or D = B, so either [AB] or [BD] has only one endpoint. No**. (Case 2)** Let [AB] = f[b',d] = g[b,c], so that [BC] = g[a,b] and [BD] = f[a',b']. If **λ > 0**, then g(t) = f(λt - λb + b'), f(t) = g(t/λ + b - b'/λ) and [BD] = f[a',b'] = g[b + (a' - b')/λ,b]. Therefore [BC] = g[a,b] and [BD] are related by Lemma 2. If **λ < 0**, then g(t) = f(λt - λb + d) and f(t) = g(t/λ + b - d/λ], so that [AD] = f[a',d] = g[b,b + (a' - d)/λ] again has endpoint g(b) = B (Contradiction). (**Case 3**) Suppose [AB] = f[b',d] = g[a,b], so that [BD] = f[a',b'] and [BC] = g[b,c]. If **λ > 0,** g(t) = f(λt - λa + b'), f(t) = g(t/λ + a - b'/λ), [AB] = f[b',d] = g[a,a + (d - b')/λ] and [BD] = f[a',b'] = g[a + (a' - b')/λ ,a]. [AB] and [BD] have (only one) endpoint g(a) in common, so g(a) = B. [AB] and [BC] have endpoint g(b) in common, so g(b) = B. Since B = g(a) = g(b), [AB] = g[a,b] has only one endpoint. No. If **λ < 0**, g(t) = f(λt - λb +b') and f(t) = g(t/λ + b - b'/λ). Then [BD] = f[a',b'] = g[b,b + (a' - b')/λ] is related to [BC] = g[b,c] by Lemma 2. (**Case 4**) Suppose that [AB] = g[b,c] = f[a',b'], so that [BD] = f[b',d] and [BC] = g[a,b]. If **λ > 0**, g(t) = f(λt - λc + b') and f(t) = g(t/λ + c - b'/λ). [AB] = g[b,c] and [BD] = f[b',d] = g[c,c + (d - b')/λ] both have endpoint g(c), so B = g(c). [AB] and [BC] both have endpoint g(b), so B = g(b). Therefore [AB] = g[b,c] has only one endpoint. If **λ < 0**, g(t) = f(λt - λb + b') and f(t) = g(t/λ + b - b'/λ). [BD] = f[b',d] = g[b + (d - b')/λ ,b] is related to [BC] = g[a,b] by Lemma 2.

Example 2.5. Let the U segments be defined on a G Space (that is, a Menger convex, finitely compact metric space which satisfies conditions of local prolongability and unique prolongability of geodesics, as discussed in (Busemann 1955), page 37). The U segments are the images of real number intervals [x, y] under locally isometric ("geodesic") maps of the real number line into the metric space, provided the endpoints at x and y are different and provided y – x is less than the period of the geodesic, if it is periodic. Theorem 5.2 of (Menzel 1996) shows this is an example.

APPENDIX 2

In (Menzel 1988 and 1996) a Greek letter with subscripts A, B represented U segment [A,B].

Theorem 1.2 of (Menzel 1988) If [AB] A [AB]* and [AB] B [AB]*, then [AB] = [AB]*.

Proof. Suppose C ∈ [AB] - [AB]*. Let [AC] be A⊆ [AB]. By Theorem 4 of this present paper [AC] A [AB]*, so one contains the other. But C ∉ [AB]*, so [AB]* ⊆ [AC]. By hypothesis [AB] B [AB]*, so that (by definition of relatedness at B) every U segment which contains one and is contained in the other must have endpoint B. Therefore [AC] has endpoint B. Since every U segment has exactly two endpoints, so B = A or C, which either contradicts the distinctness of A and B or contradicts the definition of C. By similar argument, there is no point in [AB]* - [AB].

Theorem 1.9 of (Menzel 1996). If [AB] is opposite [BC], then [AB] A ([AB] $\cup$ [BC]).

Proof. We must show that every U segment δ which contains [AB] and is contained in [AB] $\cup$ [BC] has endpoint A. Since [AB] opp [BC] , [AB] $\cup$ [BC] = [AC] which equals $\delta \cup$ [BC]. If $\delta \subseteq$ [BC], then [AB] $\subseteq$ [BC] and [BC] = [AC], so B = A (no). If [BC] $\subseteq \delta$, then δ = [AC] and q.e.d.. And if [BC] $\not\subseteq \delta$, then δ has endpoint A by Axiom 6' ([BC] does not have endpoint A).

Theorem 1.11 of (Menzel 1996). If $\alpha A\beta$, then α and β have a common opposite at A.

Proof. Assume $\alpha \subseteq \beta$. Let β' be opposite β at A. Let β'' be contained in β', A related to β' and have no common endpoints with α or β (other than A). Use the addition axiom twice to show that $\alpha \cup \beta''$ and $\beta \cup \beta''$ are U segments. Since $\beta \cap \beta''$ and $\alpha \cap \beta''$ do not contain a U segment, α and β have common opposite β''.

Theorem 1.14. The conclusion that [AC] in Axiom 6 is unique is redundant, given Axiom 8.

Proof. Suppose that [AC]* is a second such opposite. We will show that there is no point X in [CA]* – [CA]. (The proof that there is no X in [CA] – [CA]* is similar.) Given such an X, take [CX] C$\subseteq$ [CA]*, so that [CX] C [CA] (by Theorem 1.9 and Axiom 3 which is this paper's Theorem 4) and [CA] $\subseteq$ [CX]. Take [AX] opposite to [CA] in [CX] (so [AX] $\subseteq$ [CX] $\subseteq$ [CA]*). Is [AX] $\subseteq$ [AB]? (Contradicting [AB] opposite [AC]*.) Yes, because [AX] opposite [CA] and [CA] opposite [AB] implies [AX] A [AB] (by Axiom 8) and [AB] $\not\subseteq$ [AX] or else [AB] $\subseteq$ [CX] $\subseteq$ [CA]* and [CA]* $\cap$ [AB] would contain [AB] (contradiction).

APPENDIX 3

The requirement that a U segment have two endponts is inconvenient in many proofs. The addendum of (Menzel 1996) introduced "oriented sets", and (Menzel 2002) used them to define natural and real numbers and to introduce coordinates for some geodesics. Oriented sets are not used in this paper.

Definitions. An "A oriented set" or " A set" is a nonempty set of A related U segments such that (1) if $\lambda \in$ the A set, then $\{\eta : \eta \text{ A}\subseteq \lambda\} \subseteq$ the A set and (2) if all elements of the A set are A$\subseteq \sigma_0 \in \sum$, then $\sigma_0 \notin$ the A set. (E.g., $\{\eta : \eta$ A$\subseteq$ U segment [AB], proper$\}$ is an A set: **write it [[AB]}, with A first.**)

Definition (modified Eudoxus-Dedekind continuity). Let $\lambda \in \sum$ and F(λ) = {A, B}. Let T = $\{\sigma \in \sum : \sigma \text{ A}\subseteq \lambda\}$. Suppose $\forall$ partition of T into two non-empty sets T' and T'' such that every element of T’ is ⊆ every element of T”, ∃ either a λ’ ∈ T such that λ’ ⊇ every element of T’ and λ’ is ⊆ every element of T” or ∃ a λ” ∈ ∑ such that λ” B λ, λ” ⊇ every opposite (in λ) of elements of T” and λ” ⊆ every opposite (in λ) of elements of T’. Then we say **λ is *Eudoxus-Dedekind continuous.***

We need the option of either a λ’ or a λ” in the definition. (E.g., Let λ = ABAC in figure 1. Let T’ = {σ ∈ ∑ : σ A⊆ λ and σ ⊆ the loop}. (A *loop of a curve* is a section of a plane curve that is the boundary of a bounded set.) There is no λ’ (the loop ∉ ∑), but λ” is the simple curve [AC]. In the curve r = 2 + 4sinθ, T’ is {σ ∈ ∑ : σ is defined on [7π/6, θ] and θ < 11π/6}. (The loop is defined on [7π/6, 11π/6].)

Definition (modified Eudoxus-Dedekind continuity, for angles). Let λ be an angle such that G(λ) = {[AB]>, [AC]>}. Let T = {angles φ : φ [AB]>⊆ λ}}. Suppose ∀ partition of T into two non-empty sets T’ and T’’ such that every element of T’ is ⊆ every element of T”, ∃ either a λ’ ∈ T which ⊇ every element of T’ and is ⊆ in every element of T” or ∃ an angle λ” such that λ” [AC]>⊆ λ, λ” ⊇ every opposite (in λ) of elements of T” and λ” ⊆ every opposite (in λ) of elements of T’. We then say λ is **Eudoxus-Dedekind continuous.**

SOME NOTATIONS

Primitive ideas: points, surfaces, maximal-uniquely-extensible-curves (muecs), U segments and their endpoints, angles and their sides, Angles(S, A) (see introduction and first paragraph of section ANGLES.)

|**n**| is the number of elements of set n.

[AB] is a U segment with endpoints A, B.

F(σ) is the unordered set of (two) endpoints of U segment σ.

See page 2 for definitions of **U segments λ, η are A related** (**write λ A η**). If also λ ⊆ η, we may write **λ A⊆ η.**

Ray([AB], Γ)> : see definition on page 5.

Rays(S, A) is the set of all rays with endpoint A whose (U segment) elements are contained on surface S.

See definitions on pages 2 and 3 of **[AB] and [AC] are opposite (at common endpoint A)** and of **[AB] oppA [AC]** and of **[BA] opp [AC] in [BC].**

<[AB], S> : The muec in surface S which contains U segment [AB], see definition on page 5.

See definitions on page 6 of **Angles α, β are (common side) [AB]> related**, written **α [AB]> β.** If also α ⊆ β, write **α [AB]>⊆ β.**

See definitions on page 7 of **([AB]>, [AC]>) = α is opposite ([AB]>, [AD]>) = β** (at common side [AB]>) and of **α opp[AB]> β** and of **α opp β in ([AC]>, [AD]>).**

α < β means α [AB]>⊆ β, proper.

Definitions of **modified Eudoxus-Dedekind continuity for U segments and for angles** are in Appendix 3.