PREDICTION BANDS FOR RATIOS OF MEASUREMENTS WITH ENGINEERING APPLICATIONS

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ABSTRACT

Percent difference and percent error are commonly used to compare two quantities. Percent difference is applied when comparing two experimental results and percent error is used for comparing experimental or numerical results with a theoretical quantity which can be considered a correct value. The methods compared are said to be similar if the percent difference or percent error is within a specified threshold. Note that this is a point-by-point comparison, no variability in the data is taken into account, and no level of statistical confidence can be placed on the results of such comparisons. In this paper, we propose an alternative approach of comparing quantities based on prediction bands for modified percent difference or percent error in a linear regression context. The method is applied to the experimental verification of a model pressurized thick-walled cylinder with theoretical methods.

INTRODUCTION

Percent difference and percent error are routinely used in chemistry lab, physics and engineering to compare two quantities. This could be experimental versus a theoretical value, or theoretical versus numerical values, or even experimental versus experimental values. The quantities being compared are said to be similar if the percent different or percent error is say within 5% or 10%. Most of such comparisons in the literature are based on point estimates of the quantities being compared and no variability in the data are taken into consideration. Our objectives are to link percent error and percent difference to inferences for ratios of means and develop a technique for establishing a confidence band for ratios of predictions in linear regression models.
**Percent Error:** The percent error of an experimental (observed) value as compared to a true (accepted) theoretical value is defined as

\[
\text{Percent Error (PE)} = \left(\frac{|\text{Experimental} - \text{Theoretical}|}{\text{Theoretical}}\right) \times 100 \quad (1)
\]

Percent error is the absolute error expressed as a percentage of the true theoretical value.

**Percent difference:** The percent difference of two experimental values \(E_1\) and \(E_2\) is defined as

\[
\text{Percent Difference (PD)} = \left(\frac{|E_1 - E_2|}{(E_1 + E_2)/2}\right) \times 100 \quad (2)
\]

And percent difference is the absolute difference expressed as a percentage of the average of the two values being compared. If PE or PD is within 5% or 10%, the values being compared are said to be reasonably close.

Now, consider a situation in which multiple data points are available for experimental, theoretical or numerical methods. Note that one has to make a point-by-point comparison of the PE (or PD) values with a given threshold. This approach does not take into account the variability in the data and no level of statistical confidence can be placed on the results of the comparison. In the next section, statistical methods based on the modified but equivalent versions of PE and PD are proposed to efficiently make such comparisons in a regression framework.

**METHODS**

Consider regression models where the response variable (measurement) of interest \(Y\) depends on one quantitative predictor \(X\). Assume \(Y\) to be nonnegative random variable. First, we discuss the case of comparing experimental values with the corresponding theoretical values. Both theoretical and experimental measurements are assumed to be linearly related to a predictor \(X\). For a given dataset, this assumption can be verified via preliminary data exploration (example, scatterplots). Therefore, we fit the following simple linear regression models for comparing the two set of measurements.

Regression model for experimental data:

\[
Y_E = \alpha_E + \beta_E X + \varepsilon_E, \quad Y_E \sim N(\alpha_E + \beta_E X, \sigma_E^2) \quad (3)
\]
Regression model for theoretical data:
\[ Y_T = \alpha_T + \beta_T X + \varepsilon_T, \quad Y_T \sim N(\alpha_T + \beta_T X, \sigma^2_T) \]  

The two datasets can be compared in either of the following ways. If interest lies in mean response, a confidence band can be constructed for ratios of mean response from the regression lines. This can be achieved by Fieller’s (1954) point-wise confidence intervals for ratio of mean response. There is a mathematical relationship between percent error based on means (PEM) and ratio of means. Let \( \mu_E \) and \( \mu_T \) denote the means for the experimental and theoretical at a given value of the predictor \( X \), respectively. For a given threshold \( \delta \),

\[
\frac{|\mu_E - \mu_T|}{\mu_T} \leq \delta \quad \text{implies} \quad 1 - \delta \leq \frac{\mu_E}{\mu_T} \leq 1 + \delta.
\]

Therefore, inference for percent error (or relative error) based on means can be performed using standard inferences for ratio of means. But note the change in the threshold. Relative error less than or equal to 0.05 is equivalent to a ratio between 0.95 and 1.05.

For random samples of size \( n_E \) and \( n_T \) on \( (X, Y_E) \) and \( (X, Y_T) \), respectively, fit the simple linear regression models in (3) and (4), and then construct Fieller confidence interval (Djira et al. 2008) for the ratio of regression function

\[
\rho = \frac{\alpha_E + \beta_E X}{\alpha_T + \beta_T X} \quad \text{for a range of } X \text{ values.}
\]

Let \( a_E, b_E, a_T, \) and \( b_T \) respectively denote estimates of the regression coefficients \( \alpha_E, \beta_E, \alpha_T, \) and \( \beta_T \) in Equation (3) and (4). Let \( t \) be a critical point of a \( t \)-distribution with \( v \) degrees of freedom. Limits of a confidence interval for \( \rho \) are obtained by solving the equation

\[
\frac{(a_E + b_E X) - \rho(a_T + b_T X)}{\sqrt{\text{var}((a_E + b_E X) - \rho(a_T + b_T X))}} = t.
\]

This equation reduces to a quadratic equation in \( \rho \):

\[
A \rho^2 + B \rho + C = 0,
\]

where

\[
A = (a_T + b_T X)^2 - t^2[\text{var}(a_T) + X^2 \text{var}(b_T) + 2 \text{cov}(a_T, b_T)]
\]
\[
B = -2(a_E + b_E X)(a_T + b_T X)
\]
\[
C = (a_E + b_E X)^2 - t^2[\text{var}(a_E) + X^2 \text{var}(b_E) + 2 \text{cov}(a_E, b_E)]
\]
Confidence intervals for ratios of regression functions with homogeneous \((\sigma_T^2 = \sigma_E^2)\) or heterogeneous \((\sigma_T^2 \neq \sigma_E^2)\) error variances can be constructed using R software (R Development Core Team 2011) and the extension package \textit{mratios} (Dilba et al. 2007; Djira et al. 2011). In the case of homogeneous variances, \(\nu\) is the sum of the number of degrees of freedoms for the mean-square errors (MSE) of the two regressions. That is, \(\nu = (n_T - 2) + (n_E - 2)\). For heterogeneous variances, \(\nu\) can be estimated using Welch-Satterthwaite approximation and a plug-in estimate for the ratio \(\rho\).

More interestingly, a prediction band for ratios can be obtained by using in (5) the variances for predicting new observations instead of the variances for the estimated regression functions. This will yield a point-wise Fieller-type prediction band for ratios of measurements.

Similarly, percent difference for the mean of two experimental values can be related to ratio of means. And inference can be based on Fieller-type confidence band for prediction.

RESULTS

In this section, we take two numerical examples to illustrate the application of the proposed methods on experiments conducted on thick-walled cylinders (Macwan 2011) at the Department of Mechanical Engineering, South Dakota State University.

\textit{Example 1:} Experimental against theoretical

The results obtained for hoop strain from experimental and theoretical methods are compared. In the experimental method, high pressure on the inner surface of the thick-walled cylinder was generated by applying a compression load on the cylinder and hoop strain is measured on the outer surface of the thick-walled cylinder by mounting strain gages on its periphery. Lamé equations and Hooke’s law are used in the theoretical method in order to calculate hoop strain at various internal pressures. The comparison of hoop strain obtained from these two methods is done on each pressure, ranging from 500 psi to 15,000 psi. See Fig. 1 for the ratio of the experimental to theoretical values and the corresponding 95% prediction band for ratios. The band is completely below the reference horizontal line at 1, meaning that experimental values are consistently below the theoretical values. It is observed that the percent error between the two methods is within 10% for a load greater than 4,000 lb and is within 5% for a load greater than 12,000 lb.
Example 2: Experimental (Gage 1) against experimental (Gage 2)

In order to obtain desired results from the experimental method, three rosette strain gages were mounted on the thick-walled cylinder which were 120° apart from each other. The results from these strain gages should be the same or very close to each other because they were experiencing the same amount of pressure ranging from 500 psi to 15,000 psi. In Fig 2, the results from two strain gages are compared and are within the 10% threshold for loads greater than 5,000 lb.
In this paper, the percent difference and percent error for comparing two quantities are translated into inferences for ratio parameters. Fieller-type prediction bands are proposed when the measurements are linearly related to a quantitative predictor. The proposed method works for regression models with both homogeneous and heterogeneous error variances. The method accounts for the variability in the data and offers an efficient way of comparing two regressions in terms of percent error or percent difference in measurements. An alternative but probably more conservative approach for constructing prediction bands for ratios is to construct hyperbolic prediction bands for the individual regressions (for example, see Kutner et al. 2004) and then take ratios of the limits of the bands. In a future research, it would be of interest to compare the shape and coverage probability of these methods.

Figure 2. Predicted confidence band for ratio of Gage 2 to Gage 1.
LITERATURE CITED


