ILLUMINANCE FROM A POINT SOURCE IS AN OSCILLATING FUNCTION OF DISTANCE IN A HYPERSPHERICAL.universe

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ABSTRACT

The universe is to be those ordered quadruples of real numbers (x,y,z,w) ("points") in a 4-dimensional Euclidean space which satisfy the equation of the hypersphere \( x^2 + y^2 + z^2 + w^2 = 1 \), where we use the radius of the universe (called "ur") as the unit of length measurements. It is shown that the illuminance of the surface of a sphere contained in this universe, by a point light source at its center, is proportional to cosecant\(^2\) \( s \) radians, where \( s \) is the numerical value of the length of the radius of the sphere (expressed in ur units). A conclusion: a luminous body near a point in the universe antipodal to Earth may appear just as bright to an observer on Earth as if it were near Earth.

THE ARGUMENT

(Misner, Thorne & Wheeler, 1973), pages 723-724, discuss a "hyperspherical" universe in which the points are (only) those ordered quadruples of real numbers \((x,y,z,w)\) in a 4-dimensional Euclidean space which satisfy the equation \( x^2 + y^2 + z^2 + w^2 = a^2 \), and we let \( a = 1 \). The appendix discusses 4-dimensional Euclidean space, the universe and the surface area of a sphere contained in the universe (which is \( 4\pi \) \( \text{sine}^2 \) \( s \) radians) \( \text{ur}^2 \), where \( s \) is the numerical value of the length of the radius of the sphere). The reader may prefer to read the appendix next.

We will derive the illuminance law, using this formula for surface area. Assume light paths in an empty region of space are contained in great circles. The amount of energy per second passing through the surface of a sphere of radius \( s \text{ ur} \), from a point source at the center of the sphere, equals the amount of energy passing through a unit area of the sphere per second (i.e. the illuminance) times the area of the sphere. So energy per second equals illuminance times \( 4\pi \text{sine}^2 \) \( s \) radians) \( \text{ur}^2 \) and illuminance is proportional to \( 1 + \text{sine}^2 \) \( s \) radians) \( = \text{cosecant}^2 \) \( s \) radians).

From a graph of \( y = \text{csc}^2 \) \( s \) radians) we see that a luminous body receding from Earth seems (from Earth) to become progressively dimmer until \( s \) (the numerical value of its distance from Earth) is \( \pi/2 \), at which point it is 1/4 of the distance around the universe: then it becomes brighter until it approaches the antipodes \( (s = \pi) \), at which time its illuminance approaches an infinite amount. (It approaches infinity not because the luminous body is emitting an infinite amount of light, but because the surface area of the sphere around the luminous body, \( 4\pi \text{sine}^2 \) \( s \) radians), is approaching zero \( \text{ur}^2 \). When \( s = \pi \), all light
emanating from the luminous body will be concentrated on the point antipodal to it.

The identities $\sin^2(s \text{ radians}) = \sin^2\left((\pi \pm s) \text{ radians}\right) = \sin^2\left((2\pi - s) \text{ radians}\right)$ imply the illuminance on Earth from a star at distance $s$ ur from Earth is the same as that from a star of equal intensity at distance $s$ ur from the antipodes, or from a star at distance $2\pi - s$ from Earth. A star would appear equally bright (and might have different red shifts) if viewed from opposite directions (viewed from Earth). (The star might move during the difference in travelling time for the two light rays).

**SPECULATIONS**

Find a cepheid variable near the antipodes. It should appear as bright as if it were equally near Earth and as large (subtending the same angle with great-circle sides) and it might have detectable lateral motion. (One can cross many time zones in a 5 minute walk near the North Pole.) Are some “meteors” near the antipodes? Were the dinosaurs cooked by a radiant body passing through the antipodes? What would gravitational waves from a supernova do to the antipodal region?

**APPENDIX**

In our analytic 4-dimensional Euclidean geometry we call the set of all quadruples of real numbers $(x,y,z,w)$ “points”, we define the distance between points $(a,b,c,d)$ and $(x,y,z,w)$ to be $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2 + (w-d)^2}$, we define a “vector” with “initial point” $(a,b,c,d)$ and “terminal point” $(x,y,z,w)$ to be $(x-a,y-b,z-c,w-d)$, we define two vectors $(A,B,C,D)$ and $(E,F,G,H)$ to be “perpendicular” if their “scalar product” $AE+BF+CG+DH = 0$, we define the “sum” of these vectors as $(A+E,B+F,C+G,D+H)$ and we define the product of number $k$ and vector $(A,B,C,D)$ as $(kA,kB,kC,kD)$. Thus a vector $(x,y,z,w)$ may be written as a linear sum of mutually perpendicular vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ with the equation $(x,y,z,w) = x(1,0,0,0) + y((0,1,0,0) + z(0,0,1,0) + w(0,0,0,1)$. For the interested reader, (Sommerville, 1958) discusses 4-dimensional Euclidean geometry analytically and (Manning, 1914) discusses it synthetically, as does a chapter of (Wylie,1964).

As we said, the universe is to consist only of the points $(x,y,z,w)$ which satisfy $x^2 + y^2 + z^2 + w^2 = 1$. If paths are restricted to this universe, it has been shown that a shortest path between two points in the universe lies in a “great circle” ie, the circle in which the universe intersects the plane which passes through the origin $(0,0,0,0)$ and the two points. (Reducing the dimension by one, recall that one passes Labrador when flying from New York City to London: the shortest path on the surface of the Earth lies in a great circle.) We assume that light in a nearly empty region of our universe follows a path which is in a great circle. This universe may be parameterized with the equations $x = \sin \chi \sin \theta \cos \phi$, $y = \sin \chi \cos \theta$, $z = \cos \chi$, $w = \sin \chi \sin \theta \sin \phi$, because substitution of these expressions into $x^2 + y^2 + z^2 + w^2$ and the use of trigonometric identities gives 1. The universe is swept out with $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. 
$0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Taking the intersection of the universe with the plane which contains point P and the positive w axis (see diagram), we may interpret $\chi$ to be the angle between the segments ON and OP (from the equation $w = \cos \chi$), and the arc of the great circle connecting N and P has a length whose numerical value $s$ equals the numerical value of $\chi$, if $\chi$ is in radians.

We derive the formula for surface area of a sphere. Just as a circle is the set of all points in a plane which are equidistant from a fixed point (the center), so a sphere in 4-dimensional space is the set of all points in “hyperplane $Ax + By + Cz + Dw + E = 0$” which are equidistant from a center.

We first show the intersection of hyperplane $w = w_0$ and our “hypersphere” universe is a sphere, if the intersection contains more than one point. From the diagram, if $(x,y,z,w_0)$ is distance 1 ur from $(0,0,0,0)$ (ie. is in the universe), then it is constant distance $(1 - w_0^2)^{1/2}$ ur from “center” $(0,0,0,w_0)$ (ie. is on a sphere of this radius). Conversely if $(x,y,z,w_0)$ is $(1 - w_0^2)^{1/2}$ ur from $(0,0,0,w_0)$ (ie. if it is on the sphere), then it is 1 ur from $(0,0,0,0)$ (ie. it is in the universe).

See the diagram. The points in the universe whose arc-distances from the North Pole $(0,0,0,1)$ have numerical value $s_0$ lie in the hyperplane $w = w_0 = \cos(s_0 \text{ radians})$. From preceding argument, the intersection of this hyperplane and the universe is the sphere with radius $(1 - \cos^2(s_0 \text{ radians}))^{1/2}$ ur = $|\sin(s_0 \text{ radians})|$ ur and center $(0,0,0,w_0)$. With this radius, the surface area is $4\pi \sin^2(s_0 \text{ radians})$ ur$^2$. The equations of the sphere are $x^2 + y^2 + z^2 = \sin^2(s_0 \text{ radians})$ and $w = \cos(s_0 \text{ radians})$.

One may also compute the surface area of this sphere in the universe with center $(0,0,0,1)$ and radius $s_0$ ur with calculus. Since $s_0 = \chi$, substitute $s_0$ into the parametric equations of the universe to get the surface in terms of two parameters: ie. $x = \sin(s_0 \sin \theta \cos \phi)$, $y = \sin(s_0 \sin \theta \sin \phi)$, $z = \sin(s_0 \cos \theta)$, (with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$). Integrate.

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BIBLIOGRAPHY


